Introduction to Merton Jump Diffusion Model

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Abstract

This paper presents everything you need to know about Merton jump diffusion (we call it MJD) model. MJD model is one of the first beyond Black-Scholes model in the sense that it tries to capture the negative skewness and excess kurtosis of the log stock price density $\mathbb{P}\left(\ln(S_T / S_0)\right)$ by a simple addition of a compound Possion jump process. Introduction of this jump process adds three extra parameters λ, μ, and δ (to the original BS model) which give the users to control skewness and excess kurtosis of the $\mathbb{P}\left(\ln(S_T / S_0)\right)$. Merton's original approach for pricing is to use the conditional normality of MJD model and expresses the option price as conditional Black-Scholes type solution. But modern approach of its pricing is to use the Fourier transform method by Carr and Madan (1999) which is disccused in Matsuda (2004).

[1] Model Type

In this section the basic structure of MJD model is described without the derivation of the model which will be done in the next section.

MJD model is an exponential Lévy model of the form:

$$S_t = S_0 \, e^{L_t} ,$$

where the stock price process $\{S_t; 0 \le t \le T\}$ is modeled as an exponential of a Lévy process $\{L_t; 0 \le t \le T\}$. Merton's choice of the Lévy process is a Brownian motion with drift (continuous diffusion process) plus a compound Poisson process (discontinuous jump process) such that:

$$L_t = (\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t + \sum_{i=1}^{N_t} Y_i ,$$

where $\{B_t; 0 \le t \le T\}$ is a standard Brownian motion process. The term $(\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t$ is a Brownian motion with drift process and the term $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson jump process. The only difference between the Black-Scholes and the MJD is the addition of the term $\sum_{i=1}^{N_t} Y_i$. A compound Poisson jump process $\sum_{i=1}^{N_t} Y_i$ contains two sources of randomness. The first is the Poisson process dN_t with intensity (i.e. average number of jumps per unit of time) λ which causes the asset price to jump randomly (i.e. random timing). Once the asset price jumps, how much it jumps is also modeled random (i.e. random jump size). Merton assumes that log stock price jump size follows normal distribution, $\left(dx_i\right) \sim i.i.d.\ Normal(\mu, \delta^2)$:

$$f(dx_i) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\{-\frac{(dx_i - \mu)^2}{2\delta^2}\} .$$

It is assumed that these two sources of randomness are independent of each other. By introducing three extra parameters λ, μ, and δ to the original BS model, Merton JD model tries to capture the (negative) skewness and excess kurtosis of the log return density $\mathbb{P}\left(\ln\left(S_t / S_0\right)\right)$ which deviates from the BS normal log return density.

Lévy measure $\ell(dx)$ of a compound Poisson process is given by the multiplication of the intensity and the jump size density $f(dx)$:

$$\ell(dx) = \lambda f(dx) .$$

A compound Poisson process (i.e. a piecewise constant Lévy process) is called finite activity Lévy process since its Lévy measure $\ell(dx)$ is finite (i.e. the average number of jumps per unit time is finite):

$$\int_{-\infty}^{\infty} \ell(dx) = \lambda < \infty .$$

The fact that an asset price S_t is modeled as an exponential of Lévy process L_t means that its log-return $\ln(\frac{S_t}{S_0})$ is modeled as a Lévy process such that:

$$\ln(\frac{S_t}{S_0}) = L_t = (\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t + \sum_{i=1}^{N_t} Y_i .$$

Let's derive the model.

[2] Model Derivation

In MJD model, changes in the asset price consist of normal (continuous diffusion) component that is modeled by a Brownian motion with drift process and abnormal (discontinuous, i.e. jump) component that is modeled by a compound Poisson process. Asset price jumps are assumed to occur independently and identically. The probability that an asset price jumps during a small time interval dt can be written using a Poisson process dN_t as:

Pr {an asset price jumps once in dt } = Pr{ $dN_t = 1$ } $\cong$ λdt,

Pr {an asset price jumps more than once in dt } = Pr{ $dN_t \geq 2$ } $\cong$ 0,

Pr {an asset price does not jump in dt } = Pr{ $dN_t = 0$ } $\cong$ $1 - \lambda dt$,

where the parameter $\lambda \in \mathbb{R}^+$ is the intensity of the jump process (the mean number of jumps per unit of time) which is independent of time t.

Suppose in the small time interval dt the asset price jumps from S_t to $y_t S_t$ (we call y_t as absolute price jump size). So the relative price jump size (i.e. percentage change in the asset price caused by the jump) is:

$$\frac{dS_t}{S_t} = \frac{y_t S_t - S_t}{S_t} = y_t - 1,$$

where Merton assumes that the absolute price jump size y_t is a nonnegative random variables drawn from lognormal distribution, i.e. $\ln(y_t) \sim i.i.d.N(\mu, \delta^2)$. This in turn implies that $E[y_t] = e^{\mu+\frac{1}{2}\delta^2}$ and $E[(y_t - E[y_t])^2] = e^{2\mu+\delta^2}(e^{\delta^2} - 1)$. This is because if $\ln x \sim N(a,b)$, then $x \sim Lognormal(e^{a+\frac{1}{2}b^2}, e^{2a+b^2}(e^{b^2} - 1))$.

MJD dynamics of asset price which incorporates the above properties takes the SDE of the form:

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dB_t + (y_t - 1)dN_t, \tag{1}$$

where α is the instantaneous expected return on the asset, σ is the instantaneous volatility of the asset return conditional on that jump does not occur, B_t is a standard Brownian motion process, and N_t is an Poisson process with intensity λ. Standard assumption is that (B_t), (N_t), and (y_t) are independent. The relative price jump size of S_t, $y_t - 1$, is lognormally distributed with the mean $E[y_t - 1] = e^{\mu+\frac{1}{2}\delta^2} - 1 \equiv k$ and the variance $E[(y_t - 1 - E[y_t - 1])^2] = e^{2\mu+\delta^2}(e^{\delta^2} - 1)$.[1] This may be confusing to some readers, so we will repeat it again. Merton assumes that the absolute price jump size y_t is a lognormal randon variable such that:

$$(y_t) \sim i.i.d.Lognormal(e^{\mu+\frac{1}{2}\delta^2}, e^{2\mu+\delta^2}(e^{\delta^2} - 1)). \tag{2}$$

This is equivalent to saying that Merton assumes that the relative price jump size $y_t - 1$ is a lognormal random variable such that:

$$(y_t - 1) \sim i.i.d.Lognormal(k \equiv e^{\mu+\frac{1}{2}\delta^2} - 1, e^{2\mu+\delta^2}(e^{\delta^2} - 1)). \tag{3}$$

This is equivalent to saying that Merton assumes that the log price jump size $\ln y_t \equiv Y_t$ is a normal random variable such that:

$$\ln(y_t) \sim i.i.d.Normal(\mu, \delta^2). \tag{4}$$

This is equivalent to saying that Merton assumes that the log-return jump size $\ln(\frac{y_t S_t}{S_t})$ is a normal random variable such that:

[1]For random variable x, $Variance[x - 1] = Variance[x]$.

$$\ln(\frac{y_t S_t}{S_t}) = \ln(y_t) \equiv Y_t \sim i.i.d.Normal(\mu, \delta^2) . \qquad (5)$$

It is extremely important to note:

$$E[y_t - 1] = e^{\mu + \frac{1}{2}\delta^2} - 1 \equiv k \quad \neq \quad E[\ln(y_t)] = \mu ,$$

because $\ln E[y_t - 1] \neq E[\ln(y_t - 1)] = E[\ln(y_t)]$.

The expected relative price change $E[dS_t / S_t]$ from the jump part dN_t in the time interval dt is $\lambda k dt$ since $E[(y_t - 1)dN_t] = E[y_t - 1]E[dN_t] = k\lambda dt$. This is the predictable part of the jump. This is why the instantaneous expected return on the asset αdt is adjusted by $-\lambda k dt$ in the drift term of the jump-diffusion process to make the jump part an unpredictable innovation:

$$E[\frac{dS_t}{S_t}] = E[(\alpha - \lambda k)dt] + E[\sigma dB_t] + E[(y_t - 1)dN_t]$$
$$E[\frac{dS_t}{S_t}] = (\alpha - \lambda k)dt + 0 + \lambda k dt = \alpha dt .$$

Some researchers include this adjustment term for predictable part of the jump $-\lambda k dt$ in the drift term of the Brownian motion process leading to the following simpler (?) specification:

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dB_t + (y_t - 1)dN_t$$
$$B_t \sim Normal(-\frac{\lambda k t}{\sigma}, t)$$
$$E[\frac{dS_t}{S_t}] = \alpha dt + \sigma(-\frac{\lambda k dt}{\sigma}) + \lambda k dt = \alpha dt .$$

But we choose to explicitely subtract $\lambda k dt$ from the instantsneous expected return αdt because we prefer to keep B_t as a standard (zero-drift) Brownian motion process. Realize that there are two sources of randomness in MJD process. The first source is the Poisson Process dN_t which causes the asset price to jump randomly. Once the asset price jumps, how much it jumps (the jump size) is also random. It is assumed that these two sources of randomness are independent of each other.

If the asset price does not jump in small time interval dt (i.e. $dN_t = 0$), then the jump-diffusion process is simply a Brownian motion motion with drift process:

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dB_t .$$

If the asset price jumps in dt ($dN_t = 1$):

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dB_t + (y_t - 1) ,$$

the relative price jump size is $y_t - 1$. Suppose that the lognormal random drawing y_t is 0.8, the asset price falls by 20%.

Let's solve SDE of (1). From (1), MJD dynamics of an asset price is:

$$dS_t = (\alpha - \lambda k)S_t dt + \sigma S_t dB_t + (y_t - 1)S_t dN_t .$$

Cont and Tankov (2004) give the Itô formula for the jump-diffusion process as:

$$df(X_t,t) = \frac{\partial f(X_t,t)}{\partial t}dt + b_t \frac{\partial f(X_t,t)}{\partial x}dt + \frac{\sigma_t^{\ 2}}{2}\frac{\partial^2 f(X_t,t)}{\partial x^2}dt$$
$$+\sigma_t \frac{\partial f(X_t,t)}{\partial x}dB_t + [f(X_{t-} + \Delta X_t) - f(X_{t-})],$$

where b_t corresponds to the drift term and σ_t corresponds to the volatility term of a jump-diffusion process $X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \sum_{i=1}^{N_t} \Delta X_i$. By applying this:

$$d\ln S_t = \frac{\partial \ln S_t}{\partial t}dt + (\alpha - \lambda k)S_t \frac{\partial \ln S_t}{\partial S_t}dt + \frac{\sigma^2 S_t^{\ 2}}{2}\frac{\partial^2 \ln S_t}{\partial S_t^{\ 2}}dt$$
$$+\sigma S_t \frac{\partial \ln S_t}{\partial S_t}dB_t + [\ln y_t S_t - \ln S_t]$$

$$d\ln S_t = (\alpha - \lambda k)S_t \frac{1}{S_t}dt + \frac{\sigma^2 S_t^{\ 2}}{2}\left(-\frac{1}{S_t^{\ 2}}\right)dt + \sigma S_t \frac{1}{S_t}dB_t + [\ln y_t + \ln S_t - \ln S_t]$$

$$d\ln S_t = (\alpha - \lambda k)dt - \frac{\sigma^2}{2}dt + \sigma_t dB_t + \ln y_t$$

$$\ln S_t - \ln S_0 = (\alpha - \frac{\sigma^2}{2} - \lambda k)(t-0) + \sigma_t (B_t - B_0) + \sum_{i=1}^{N_t} \ln y_i$$

$$\ln S_t = \ln S_0 + (\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma_t B_t + \sum_{i=1}^{N_t} \ln y_i$$

$$\exp\left(\ln S_t\right) = \exp\left\{\ln S_0 + (\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma_t B_t + \sum_{i=1}^{N_t} \ln y_i\right\}$$

$$S_t = S_0 \exp\left\{\left(\alpha - \frac{\sigma^2}{2} - \lambda k\right)t + \sigma_t B_t\right\} \exp\left(\sum_{i=1}^{N_t} \ln y_i\right)$$

$$S_t = S_0 \exp[(\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t] \prod_{i=1}^{N_t} y_i ,$$

or alternatively as:

$$S_t = S_0 \exp[(\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t + \sum_{i=1}^{N_t} \ln y_i] .$$

Using the previous definition of the log price (return) jump size $\ln(y_t) \equiv Y_t$:

$$S_t = S_0 \exp[(\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t + \sum_{i=1}^{N_t} Y_i] . \tag{6}$$

This means that the asset price process $\{S_t ; 0 \le t \le T\}$ is modeled as an exponential Lévy model of the form:

$$S_t = S_0 \, e^{L_t} ,$$

where X_t is a Lévy process which is categorized as a Brownian motion with drift (continuous part) plus a compound Poisson process (jump part) such that:

$$L_t = (\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t + \sum_{i=1}^{N_t} Y_i .$$

In other words, log-return $\ln(\frac{S_t}{S_0})$ is modeled as a Lévy process such that:

$$\ln(\frac{S_t}{S_0}) = L_t = (\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t + \sum_{i=1}^{N_t} Y_i .$$

Note that the compound Poisson jump process $\prod_{i=1}^{N_t} y_i = 1$ (in absolute price scale) or $\sum_{i=1}^{N_t} \ln y_i = \sum_{i=1}^{N_t} Y_i = 0$ (in log price scale) if $N_t = 0$ (i.e. no jumps between time 0 and t) or positive and negative jumps cancel each other out.

In the Black-Scholes case, log return $\ln(S_t / S_0)$ is normally distributed:

$$S_t = S_0 \exp[(\alpha - \frac{\sigma^2}{2})t + \sigma B_t]$$
$$\ln(\frac{S_t}{S_0}) \sim Normal[(\alpha - \frac{\sigma^2}{2})t, \sigma^2 t].$$

But in MJD case, the existence of compound Poisson jump process $\sum_{i=1}^{N_t} Y_i$ makes log return non-normal. In Merton's case the simple distributional assumption about the log return jump size $(Y_i) \sim N(\mu, \delta^2)$ enables the probability density of log return $x_t = \ln(S_t / S_0)$ to be obtained as a quickly converging series of the following form:

$$\mathbb{P}(x_t \in A) = \sum_{i=0}^{\infty} \mathbb{P}(N_t = i)\mathbb{P}(x_t \in A \mid N_t = i)$$
$$\mathbb{P}(x_t) = \sum_{i=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} N(x_t; (\alpha - \frac{\sigma^2}{2} - \lambda k)t + i\mu, \sigma^2 t + i\delta^2) \tag{7}$$

where $N(x_t; (\alpha - \frac{\sigma^2}{2} - \lambda k)t + i\mu, \sigma^2 t + i\delta^2)$

$$= \frac{1}{\sqrt{2\pi(\sigma^2 t + i\delta^2)}} \exp[-\frac{\left[x_t - \left\{\left(\alpha - \frac{\sigma^2}{2} - \lambda k\right)t + i\mu\right\}\right]^2}{2(\sigma^2 t + i\delta^2)}].$$

The term $\mathbb{P}(N_t = i) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}$ is the probability that the asset price jumps i times during the time interval of length t. And $\mathbb{P}(x_t \in A \mid N_t = i) = N(x_t; (\alpha - \frac{\sigma^2}{2} - \lambda k)t + i\mu, \sigma^2 t + i\delta^2)$ is the Black-Scholes normal density of log-return assuming that the asset price jumps i times in the time interval of t. Therefore, the log-return density in the MJD model can be interpreted as the weighted average of the Black-Scholes normal density by the probability that the asset price jumps i times.

By Fourier transforming the Merton log-return density function with FT parameters $(a, b,) = (1,1)$, its characteristic function is calculated as:

$$\phi(\omega) = \int_{-\infty}^{\infty} \exp\left(i\omega x_t\right)\mathbb{P}(x_t)dx_t$$

$$= \exp\left[\lambda t \exp\left\{ \frac{1}{2} \omega \left(2i\mu - \delta^2 \omega \right) \right\} - \lambda t \left(1 + i\omega k \right) - \frac{1}{2} t\omega \left\{ -2i\alpha + \sigma^2 \left(i + \omega \right) \right\} \right].$$

After simplification:

$$\phi(\omega) = \exp\left[t\psi(\omega) \right]$$

with the characteristic exponent (cumulant generating function):

$$\psi(\omega) = \lambda \left\{ \exp\left(i\omega\mu - \frac{\delta^2\omega^2}{2} \right) - 1 \right\} + i\omega\left(\alpha - \frac{\sigma^2}{2} - \lambda k \right) - \frac{\sigma^2\omega^2}{2}, \tag{8}$$

where $k \equiv e^{\mu + \frac{1}{2}\delta^2} - 1$. The characteristic exponent (8) can be alternatively obtained by substituting the Lévy measure of the MJD model:

$$\ell(dx) = \frac{\lambda}{\sqrt{2\pi\delta^2}} \exp\left\{ -\frac{\left(dx - \mu \right)^2}{2\delta^2} \right\} = \lambda f(dx)$$
$$f(dx) \sim N\left(\mu, \delta^2 \right)$$

into the Lévy-Khinchin representation of the finite variation type (read Matsuda (2004)):

$$\psi(\omega) = ib\omega - \frac{\sigma^2\omega^2}{2} + \int_{-\infty}^{\infty} \left\{ \exp(i\omega x) - 1 \right\} \ell(dx)$$
$$\psi(\omega) = ib\omega - \frac{\sigma^2\omega^2}{2} + \int_{-\infty}^{\infty} \left\{ \exp(i\omega x) - 1 \right\} \lambda f(dx)$$
$$\psi(\omega) = ib\omega - \frac{\sigma^2\omega^2}{2} + \lambda \int_{-\infty}^{\infty} \left\{ \exp(i\omega x) - 1 \right\} f(dx)$$
$$\psi(\omega) = ib\omega - \frac{\sigma^2\omega^2}{2} + \lambda \left\{ \int_{-\infty}^{\infty} e^{i\omega x} f(dx) - \int_{-\infty}^{\infty} f(dx) \right\}$$

Note that $\int_{-\infty}^{\infty} e^{i\omega x} f(dx)$ is the characteristic function of $f(dx)$:

$$\int_{-\infty}^{\infty} e^{i\omega x} f(dx) = \exp\left(i\mu\omega - \frac{\delta^2\omega^2}{2} \right).$$

Therefore:

$$\psi(\omega) = ib\omega - \frac{\sigma^2\omega^2}{2} + \lambda\left\{\exp\left(i\mu\omega - \frac{\delta^2\omega^2}{2}\right) - 1\right\},$$

where $b = \alpha - \frac{\sigma^2}{2} - \lambda k$. This corresponds to (8). Characteristic exponent (8) generates cumulants as follows:

$$\begin{aligned}
cumulant_1 &= \alpha - \frac{\sigma^2}{2} - \lambda k + \lambda\mu, \\
cumulant_2 &= \sigma^2 + \lambda\delta^2 + \lambda\mu^2, \\
cumulant_3 &= \lambda(3\delta^2\mu + \mu^3), \\
cumulant_4 &= \lambda(3\delta^4 + 6\mu^2\delta^2 + \mu^4).
\end{aligned}$$

Annualized (per unit of time) mean, variance, skewness, and excess kurtosis of the log-return density $\mathbb{P}(x_t)$ are computed from above cumulants as follows:

$$\begin{aligned}
E[x_t] &= cumulant_1 = \alpha - \frac{\sigma^2}{2} - \lambda\left(e^{\mu + \frac{1}{2}\delta^2} - 1\right) + \lambda\mu \\
Variance[x_t] &= cumulant_2 = \sigma^2 + \lambda\delta^2 + \lambda\mu^2 \\
Skewness[x_t] &= \frac{cumulant_3}{(cumulant_2)^{3/2}} = \frac{\lambda(3\delta^2\mu + \mu^3)}{(\sigma^2 + \lambda\delta^2 + \lambda\mu^2)^{3/2}} \\
Excess\ Kurtosis[x_t] &= \frac{cumulant_4}{(cumulant_2)^{4/2}} = \frac{\lambda(3\delta^4 + 6\mu^2\delta^2 + \mu^4)}{(\sigma^2 + \lambda\delta^2 + \lambda\mu^2)^2}.
\end{aligned} \tag{9}$$

We can observe several interesting properties of Merton's log-return density $\mathbb{P}(x_t)$. Firstly, the sign of μ which is the expected log-return jump size, $E[Y_t] = \mu$, determines the sign of skewness. The log-return density $\mathbb{P}(x_t)$ is negatively skewed if $\mu < 0$ and it is symmetric if $\mu = 0$ as illustrated in Figure 1.

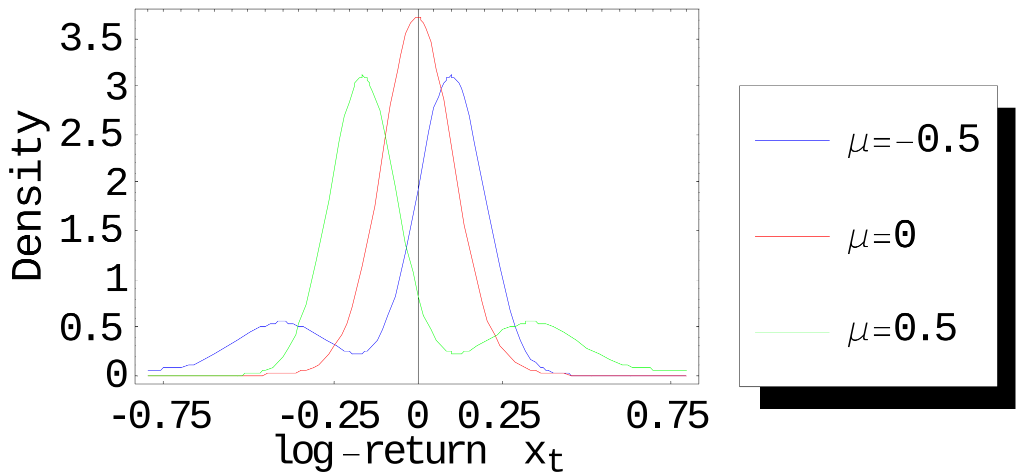


Figure 1: Merton's Log-Return Density for Different Values of μ. $\mu = -0.5$ in blue, $\mu = 0$ in red, and $\mu = 0.5$ in green. Parameters fixed are $\tau = 0.25$, $\alpha = 0.03$, $\sigma = 0.2$, $\lambda = 1$, and $\delta = 0.1$.

Table 1
Annualized Moments of Merton's Log-Return Density in Figure 1

Model	Mean	Standard Deviation	Skewness	Excess Kurtosis
$\mu = -0.5$	-0.0996	0.548	-0.852	0.864
$\mu = 0$	0.005	0.3742	0	0.12
$\mu = 0.5$	-0.147	0.5477	0.852	0.864

Secondly, larger value of intensity λ (which means that jumps are expected to occur more frequently) makes the density fatter-tailed as illustrated in Figure 2. Note that the excess kurtosis in the case $\lambda = 100$ is much smaller than in the case $\lambda = 1$ or $\lambda = 10$. This is because excess kurtosis is a standardized measure (by standard deviation).

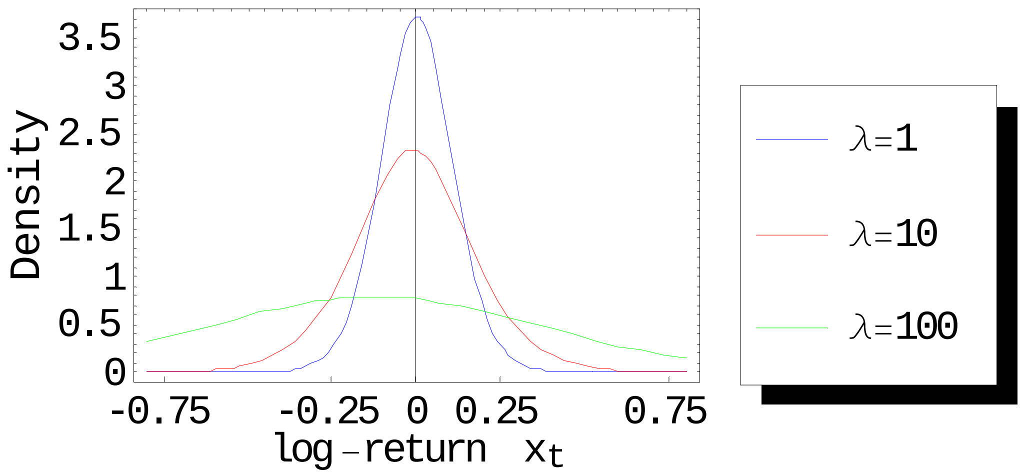


Figure 2: Merton's Log-Return Density for Different Values of Intensity λ. $\lambda = 1$ in blue, $\lambda = 10$ in red, and $\lambda = 100$ in green. Parameters fixed are $\tau = 0.25$, $\alpha = 0.03$, $\sigma = 0.2$, $\mu = 0$, and $\delta = 0.1$.

Table 2
Annualized Moments of Merton's Log-Return Density in Figure 2

Model	Mean	Standard Deviation	Skewness	Excess Kurtosis
$\lambda = 1$	0.00499	0.2236	0	0.12
$\lambda = 10$	-0.04012	0.3742	0	0.1531
$\lambda = 100$	-0.49125	1.0198	0	0.0277

Also note that Merton's log-return density has higher peak and fatter tails (more leptokurtic) when matched to the Black-Scholes normal counterpart as illustrated in Figure 9.3.

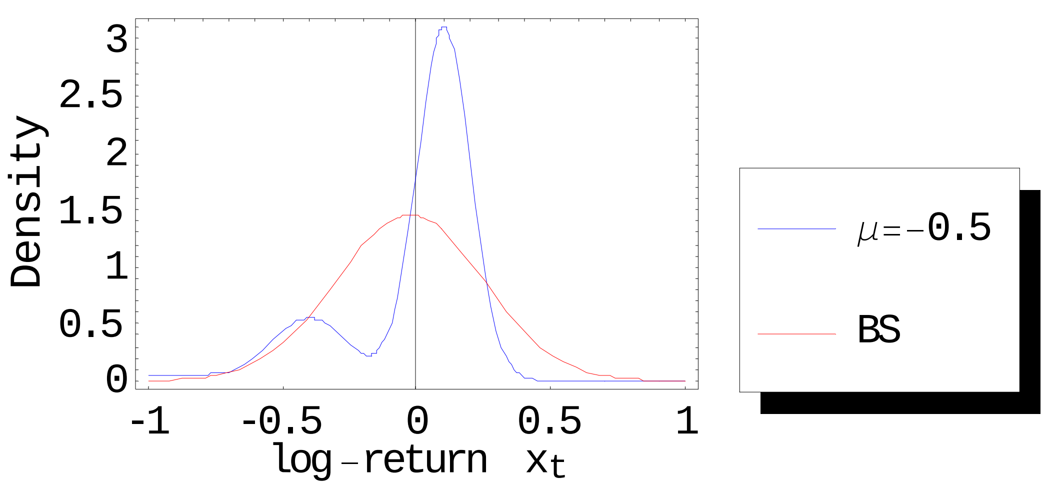


Figure 3: Merton Log-Return Density vs. Black-Scholes Log-Return Density (Normal). Parameters fixed for the Merton (in blue) are $\tau = 0.25$, $\alpha = 0.03$, $\sigma = 0.2$, $\lambda = 1$, $\mu = -0.5$, and $\delta = 0.1$. Black-Scholes normal log-return density is plotted (in red) by matching the mean and variance to the Merton.

Table 3
Annualized Moments of Merton vs. Black-Scholes Log-Return Density in Figure 3

Model	Mean	Standard Deviation	Skewness	Excess Kurtosis
Merton with $\mu = -0.5$	-0.0996	0.548	-0.852	0.864
Black-Scholes	-0.0996	0.548	0	0

[3] Log Stock Price Process for Merton Jump-Diffusion Model

Log stock price dynamics can be obtained from the equation (6) as:

$$\ln S_t = \ln S_0 + \left(\alpha - \frac{\sigma^2}{2} - \lambda k \right) t + \sigma B_t + \sum_{i=1}^{N_t} Y_i . \tag{10}$$

Probability density of log stock price $\ln S_t$ is obtained as a quickly converging series of the following form (i.e. conditionally normal):

$$\mathbb{P}(\ln S_t \in A) = \sum_{i=0}^{\infty} \mathbb{P}(N_t = i) \mathbb{P}(\ln S_t \in A \big| N_t = i)$$

$$\mathbb{P}(\ln S_t) = \sum_{i=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^i}{i!} N\left(\ln S_t ; \ln S_0 + \left(\alpha - \frac{\sigma^2}{2} - \lambda k \right) t + i\mu, \sigma^2 t + i\delta^2 \right), \tag{11}$$

where:

$$N\left(\ln S_t ; \ln S_0 + \left(\alpha - \frac{\sigma^2}{2} - \lambda k \right) t + i\mu, \sigma^2 t + i\delta^2 \right)$$

$$= \frac{1}{\sqrt{2\pi \left(\sigma^2 t + i\delta^2 \right)}} \exp\left[- \frac{\left\{ \ln S_t - \left(\ln S_0 + \left(\alpha - \frac{\sigma^2}{2} - \lambda k \right) t + i\mu \right) \right\}^2}{2\left(\sigma^2 t + i\delta^2 \right)} \right].$$

By Fourier transforming (11) with FT parameters $(a,b,)=(1,1)$, its characteristic function is calculated as:

$$\begin{aligned}\phi(\omega) &= \int_{-\infty}^{\infty} \exp\left(i\omega \ln S_t\right)\mathbb{P}(\ln S_t)d\ln S_t \\ &= \exp\left[\lambda t\left(\exp(i\mu\omega - \frac{\delta^2\omega^2}{2}) - 1\right) + i\omega\left(\ln S_0 + (\alpha - \frac{\sigma^2}{2} - \lambda k)t\right) - \frac{\sigma^2\omega^2}{2}t\right],\end{aligned} \quad (12)$$

where $k = e^{\mu + \frac{\delta^2}{2}} - 1$.

[4] Lévy Measure for Merton Jump-Diffusion Model

Lévy measure $\ell(dx)$ of a compound Poisson process is given by the multiplication of the intensity and the jump size density $f(dx)$:

$$\ell(dx) = \lambda f(dx).$$

The Lévy measure $\ell(dx)$ represents the arrival rate (i.e. total intensity) of jumps of sizes $[x, x+dx]$. In other words, we can interpret the Lévy measure $\ell(dx)$ of a compound Poisson process as the measure of the average number of jumps per unit of time. Lévy measure is a positive measure on $\mathbb{R}$, but it is not a probability measure since its total mass λ (in the compound Poisson case) does not have to equal1:

$$\int \ell(dx) = \lambda \in \mathbb{R}^+.$$

A Poisson process and a compound Poisson process (i.e. a piecewise constant Lévy process) are called finite activity Lévy processes since their Lévy measures $\ell(dx)$ are finite (i.e. the average number of jumps per unit time is finite):

$$\int_{-\infty}^{\infty} \ell(dx) < \infty.$$

In Merton jump-diffusion case, the log-return jump size is $\left(dx_i\right) \sim i.i.d.\ Normal(\mu, \delta^2)$:

$$f(dx_i) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\{-\frac{(dx_i - \mu)^2}{2\delta^2}\}.$$

Therefore, the Lévy measure $\ell(dx)$ for Merton case can be expressed as:

$$\ell(dx) = \lambda f(dx) = \frac{\lambda}{\sqrt{2\pi\delta^2}} \exp\{-\frac{(dx - \mu)^2}{2\delta^2}\}. \quad (13)$$

An example of Lévy measure $\ell(dx)$ for the log-return $x_t = \ln(S_t / S_0)$ in MJD model is plotted in Figure 5. Each Lévy measure is symmetric (i.e. $\mu = 0$ is used) with total mass 1, 2, and 4 respectively.

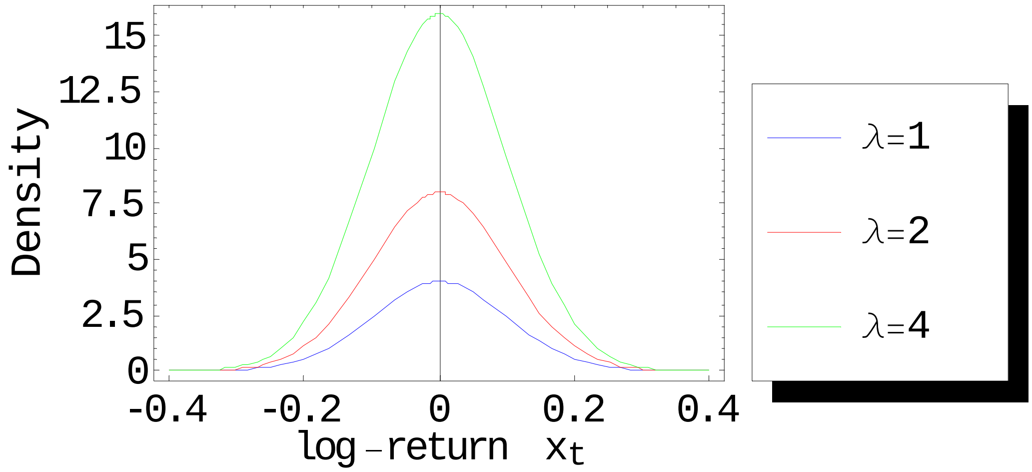


Figure 5: Lévy Measures $\ell(dx)$ for the Log-Return $x_t = \ln(S_t / S_0)$ in MJD Model for Different Values of Intensity λ. Parameters used are $\mu = 0$ and $\delta = 0.1$.

[5] Option Pricing: PDE Approach by Hedging

Consider a portfolio P of the one long option position $V(S,t)$ on the underlying asset S written at time t and a short position of the underlying asset in quantity Δ to derive option pricing functions in the presence of jumps:

$$P_t = V(S_t, t) - \Delta S_t . \tag{14}$$

Portfolio value changes by in a very short period of time:

$$dP_t = dV(S_t, t) - \Delta dS_t . \tag{15}$$

MJD dynamics of an asset price is given by equation (1) in the differential form as:

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dB_t + (y_t - 1)dN_t ,$$
$$dS_t = (\alpha - \lambda k)S_t dt + \sigma S_t dB_t + (y_t - 1)S_t dN_t . \tag{16}$$

Itô formula for the jump-diffusion process is given as (Cont and Tankov (2004)):

$$df(X_t,t)=\frac{\partial f(X_t,t)}{\partial t}dt+b_t\frac{\partial f(X_t,t)}{\partial x}dt+\frac{\sigma_t^{\,2}}{2}\frac{\partial^2 f(X_t,t)}{\partial x^2}dt$$
$$+\sigma_t\frac{\partial f(X_t,t)}{\partial x}dB_t+[f(X_{t-}+\Delta X_t)-f(X_{t-})],$$

where b_t corresponds to the drift term and σ_t corresponds to the volatility term of a jump-diffusion process $X_t=X_0+\int_0^t b_s ds+\int_0^t \sigma_s dB_s+\sum_{i=1}^{N_t}\Delta X_i$. Apply this to our case of option price function $V(S,t)$:

$$dV(S_t,t)=\frac{\partial V}{\partial t}dt+(\alpha-\lambda k)S_t\frac{\partial V}{\partial S_t}dt+\frac{\sigma^2 S_t^{\,2}}{2}\frac{\partial^2 V}{\partial S_t^{\,2}}dt+\sigma S_t\frac{\partial V}{\partial S_t}dB_t$$
$$+[V(y_tS_t,t)-V(S_t,t)]dN_t. \tag{17}$$

The term $[V(y_tS_t,t)-V(S_t,t)]dN_t$ describes the difference in the option value when a jump occurs. Now the change in the portfolio value can be expressed as by substituting (16) and (17) into (15):

$$dP_t=dV(S_t,t)-\Delta dS_t$$
$$dP_t=\frac{\partial V}{\partial t}dt+(\alpha-\lambda k)S_t\frac{\partial V}{\partial S_t}dt+\frac{\sigma^2 S_t^{\,2}}{2}\frac{\partial^2 V}{\partial S_t^{\,2}}dt+\sigma S_t\frac{\partial V}{\partial S_t}dB_t+[V(y_tS_t,t)-V(S_t,t)]dN_t$$
$$-\Delta\{(\alpha-\lambda k)S_tdt+\sigma S_tdB_t+(y_t-1)S_tdN_t\}$$
$$dP_t=\left\{\frac{\partial V}{\partial t}+(\alpha-\lambda k)S_t\frac{\partial V}{\partial S_t}+\frac{\sigma^2 S_t^{\,2}}{2}\frac{\partial^2 V}{\partial S_t^{\,2}}-\Delta(\alpha-\lambda k)S_t\right\}dt+\left(\sigma S_t\frac{\partial V}{\partial S_t}-\Delta\sigma S_t\right)dB_t$$
$$+\left\{V(y_tS_t,t)-V(S_t,t)-\Delta(y_t-1)S_t\right\}dN_t. \tag{18}$$

If there is no jump between time 0 and t (i.e. $dN_t=0$), the problem reduces to Black-Scholes case in which setting $\Delta=\partial V/\partial S_t$ makes the portfolio risk-free leading to the following (i.e. the randomness dB_t has been eliminated):

$$dP_t=\left\{\frac{\partial V}{\partial t}+(\alpha-\lambda k)S_t\frac{\partial V}{\partial S_t}+\frac{\sigma^2 S_t^{\,2}}{2}\frac{\partial^2 V}{\partial S_t^{\,2}}-\frac{\partial V}{\partial S_t}(\alpha-\lambda k)S_t\right\}dt+\left(\sigma S_t\frac{\partial V}{\partial S_t}-\frac{\partial V}{\partial S_t}\sigma S_t\right)dB_t$$
$$dP_t=\left\{\frac{\partial V}{\partial t}+\frac{\sigma^2 S_t^{\,2}}{2}\frac{\partial^2 V}{\partial S_t^{\,2}}\right\}dt.$$

This in turn means that if there is a jump between time 0 and t (i.e. $dN_t\neq 0$), setting $\Delta=\partial V/\partial S_t$ does not eliminate the risk. Suppose we decided to hedge the randomness caused by diffusion part dB_t in the underlying asset price (which are always present) and not to hedge the randomness caused by jumps dN_t (which occur infrequently) by setting $\Delta = \partial V / \partial S_t$. Then, the change in the value of the portfolio is given by from equation (9.18):

$$dP_t = \left\{ \frac{\partial V}{\partial t} + (\alpha - \lambda k) S_t \frac{\partial V}{\partial S_t} + \frac{\sigma^2 S_t^{\,2}}{2} \frac{\partial^2 V}{\partial S_t^{\,2}} - \frac{\partial V}{\partial S_t} (\alpha - \lambda k) S_t \right\} dt + \left(\sigma S_t \frac{\partial V}{\partial S_t} - \frac{\partial V}{\partial S_t} \sigma S_t \right) dB_t$$
$$+ \left\{ V(y_t S_t, t) - V(S_t, t) - \frac{\partial V}{\partial S_t} (y_t - 1) S_t \right\} dN_t ,$$
$$dP_t = \left\{ \frac{\partial V}{\partial t} + \frac{\sigma^2 S_t^{\,2}}{2} \frac{\partial^2 V}{\partial S_t^{\,2}} \right\} dt + \left\{ V(y_t S_t, t) - V(S_t, t) - \frac{\partial V}{\partial S_t} (y_t - 1) S_t \right\} dN_t . \quad (19)$$

Merton argues that the jump component (dN_t) of the asset price process S_t is uncorrelated with the market as a whole. Then, the risk of jump is diversifiable (non-systematic) and it should earn no risk premium. Therefore, the portfolio is expected to grow at the risk-free interest rate r:

$$E[dP_t] = r P_t dt . \quad (20)$$

After substitution of (14) and (19) into (20) by setting $\Delta = \partial V / \partial S_t$:

$$E[\left\{ \frac{\partial V}{\partial t} + \frac{\sigma^2 S_t^{\,2}}{2} \frac{\partial^2 V}{\partial S_t^{\,2}} \right\} dt + \left\{ V(y_t S_t, t) - V(S_t, t) - \frac{\partial V}{\partial S_t} (y_t - 1) S_t \right\} dN_t] = r\{V(S_t, t) - \Delta S_t\} dt$$
$$(\frac{\partial V}{\partial t} + \frac{\sigma^2 S_t^{\,2}}{2} \frac{\partial^2 V}{\partial S_t^{\,2}}) dt + E[V(y_t S_t, t) - V(S_t, t) - \frac{\partial V}{\partial S_t} (y_t - 1) S_t] E[dN_t] = r\{V(S_t, t) - \frac{\partial V}{\partial S_t} S_t\} dt$$
$$(\frac{\partial V}{\partial t} + \frac{\sigma^2 S_t^{\,2}}{2} \frac{\partial^2 V}{\partial S_t^{\,2}}) dt + E[V(y_t S_t, t) - V(S_t, t) - \frac{\partial V}{\partial S_t} (y_t - 1) S_t] \lambda dt = r\{V(S_t, t) - \frac{\partial V}{\partial S_t} S_t\} dt$$
$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S_t^{\,2}}{2} \frac{\partial^2 V}{\partial S_t^{\,2}} + \lambda E[V(y_t S_t, t) - V(S_t, t) - \frac{\partial V}{\partial S_t} (y_t - 1) S_t] = r\{V(S_t, t) - \frac{\partial V}{\partial S_t} S_t\}$$

Thus, the MJD counterpart of Black-Scholes PDE is:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S_t^{\,2}}{2} \frac{\partial^2 V}{\partial S_t^{\,2}} + r S_t \frac{\partial V}{\partial S_t} - rV + \lambda E[V(y_t S_t, t) - V(S_t, t)] - \lambda S_t \frac{\partial V}{\partial S_t} E[y_t - 1] = 0 . \quad (21)$$

where the term $E[V(y_t S_t, t) - V(S_t, t)]$ involves the expectation operator and $E[y_t - 1] = e^{\mu + \frac{1}{2}\delta^2} - 1 \equiv k$ (which is the mean of relative asset price jump size). Obviously, if jump is not expected to occur (i.e. $\lambda = 0$), this reduces to Black-Scholes PDE:[2]

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S_t^{\ 2}}{2} \frac{\partial^2 V}{\partial S_t^{\ 2}} + r S_t \frac{\partial V}{\partial S_t} - rV = 0 .$$

Merton's simple assumption that the absolute price jump size is lognormally distributed (i.e. the log-return jump size is normally distributed, $Y_t \equiv \ln(y_t) \sim N(\mu, \delta^2)$) makes it possible to solve the jump-diffusion PDE to obtain the following price function of European vanilla options as a quickly converging series of the form:

$$\sum_{i=0}^{\infty} \frac{e^{-\bar{\lambda}\tau} (\bar{\lambda}\tau)^i}{i!} V_{BS}(S_t, \tau = T - t, \sigma_i, r_i) , \qquad (22)$$

where $\bar{\lambda} = \lambda(1+k) = \lambda e^{\mu + \frac{1}{2}\delta^2}$,

$$\sigma_i^2 = \sigma^2 + \frac{i\delta^2}{\tau} ,$$

$$r_i = r - \lambda k + \frac{i \ln(1+k)}{\tau} = r - \lambda(e^{\mu + \frac{1}{2}\delta^2} - 1) + \frac{i(\mu + \frac{1}{2}\delta^2)}{\tau} ,$$

and V_{BS} is the Black-Scholes price without jumps.

Thus, MJD option price can be interpreted as the weighted average of the Black-Scholes price conditional on that the underlying asset price jumps i times to the expiry with weights being the probability that the underlying jumps i times to the expiry.

[6] Option Pricing: Martingale Approach

Let $\{B_t; 0 \le t \le T\}$ be a standard Brownian motion process on a space $(\Omega, \mathcal{F}, \mathbb{P})$. Under actual probability measure $\mathbb{P}$, the dynamics of MJD asset price process is given by equation (6) in the integral form:

$$S_t = S_0 \exp[(\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t + \sum_{k=1}^{N_t} Y_k] .$$

[2] This equation not only contains local derivatives but also links together option values at discontinuous values in S. This is called non-local nature.

We changed the index from $\sum_{i=1}^{N_t} Y_i$ to $\sum_{k=1}^{N_t} Y_k$. This is trivial but readers will find the reason soon. MJD model is an example of an incomplete model because there are many equivalent martingale risk-neutral measures $\mathbb{Q} \sim \mathbb{P}$ under which the discounted asset price process $\{e^{-rt}S_t; 0 \le t \le T\}$ becomes a martingale. Merton finds his equivalent martingale risk-neutral measure $\mathbb{Q}_M \sim \mathbb{P}$ by changing the drift of the Brownian motion process while keeping the other parts (most important is the jump measure, i.e. the distribution of jump times and jump sizes) unchanged:

$$S_t = S_0 \exp[(r - \frac{\sigma^2}{2} - \lambda k)t + \sigma B_t^{\mathbb{Q}_M} + \sum_{k=1}^{N_t} Y_k] \quad \text{under } \mathbb{Q}_M . \tag{23}$$

Note that $B_t^{\mathbb{Q}_M}$ is a standard Brownian motion process on $(\Omega, \mathcal{F}, \mathbb{Q}_M)$ and the process $\{e^{-rt}S_t; 0 \le t \le T\}$ is a martingale under $\mathbb{Q}_M$. Then, a European option price $V^{Merton}(t, S_t)$ with payoff function $H(S_T)$ is calculated as:

$$V^{Merton}(t, S_t) = e^{-r(T-t)} E^{\mathbb{Q}_M}[H(S_T) | \mathcal{F}_t]. \tag{24}$$

Standard assumption is $\mathcal{F}_t = S_t$, thus:

$$V^{Merton}(t, S_t) = e^{-r(T-t)} E^{\mathbb{Q}_M}[H(S_t \exp[(r - \frac{\sigma^2}{2} - \lambda k)(T-t) + \sigma B_{T-t}^{\mathbb{Q}_M} + \sum_{k=1}^{N_{T-t}} Y_k]) | S_t]$$
$$V^{Merton}(t, S_t) = e^{-r(T-t)} E^{\mathbb{Q}_M}[H(S_t \exp[(r - \frac{\sigma^2}{2} - \lambda k)(T-t) + \sigma B_{T-t}^{\mathbb{Q}_M} + \sum_{k=1}^{N_{T-t}} Y_k])]. \tag{25}$$

Poisson counter is (we would like to use index i for the number of jumps):

$$N_{T-t} = 0, 1, 2, ... \equiv i .$$

And the compound Poisson process is distributed as:

$$\sum_{k=1}^{N_{T-t}} Y_k \sim Normal(i\mu, i\delta^2) .$$

Thus, $V^{Merton}(t, S_t)$ can be expressed as from equation (25) (i.e. by conditioning on i):

$$\begin{aligned} &V^{Merton}(t, S_t) \\ &= e^{-r(T-t)} \sum_{i \ge 0} \mathbb{Q}_M(N_{T-t} = i) E^{\mathbb{Q}_M}[H(S_t \exp[(r - \frac{\sigma^2}{2} - \lambda k)(T-t) + \sigma B_{T-t}^{\mathbb{Q}_M} + \sum_{k=1}^{i} Y_k])]. \end{aligned}$$

Use $\tau = T - t$:

$$V^{Merton}(t, S_t) = e^{-r\tau}\sum_{i\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}E^{\mathbb{Q}_M}\left[H\left(S_t \exp[\{r - \frac{\sigma^2}{2} - \lambda(e^{\mu+\delta^2/2} - 1)\}\tau + \sigma B_\tau^{\mathbb{Q}_M} + \sum_{k=1}^{i} Y_k]\right)\right]. \qquad (26)$$

Inside the exponential function is normally distributed:

$$\{r - \frac{\sigma^2}{2} - \lambda(e^{\mu+\delta^2/2} - 1)\}\tau + \sigma B_\tau^{\mathbb{Q}_M} + \sum_{k=1}^{i} Y_k \sim Normal\left(\{r - \frac{\sigma^2}{2} - \lambda(e^{\mu+\delta^2/2} - 1)\}\tau + i\mu, \sigma^2\tau + i\delta^2\right).$$

Rewrite it so that its distribution remains the same:

$$\{r - \frac{\sigma^2}{2} - \lambda(e^{\mu+\delta^2/2} - 1)\}\tau + i\mu + \sqrt{\frac{\sigma^2\tau + i\delta^2}{\tau}}B_\tau^{\mathbb{Q}_M} \sim Normal\left(\{r - \frac{\sigma^2}{2} - \lambda(e^{\mu+\delta^2/2} - 1)\}\tau + i\mu, \sigma^2\tau + i\delta^2\right).$$

Now we can rewrite equation (24) as (we can do this operation because a normal density is uniquely determined by only two parameters: its mean and variance):

$$V^{Merton}(t, S_t) = e^{-r\tau}\sum_{i\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}E^{\mathbb{Q}_M}\left[H\left(S_t \exp[\{r - \frac{\sigma^2}{2} - \lambda(e^{\mu+\delta^2/2} - 1)\}\tau + i\mu + \sqrt{\frac{\sigma^2\tau + i\delta^2}{\tau}}B_\tau^{\mathbb{Q}_M}]\right)\right].$$

We can always add $(\frac{i\delta^2}{2\tau} - \frac{i\delta^2}{2\tau}) = 0$ inside the exponential function:

$$V^{Merton}(t, S_t) = e^{-r\tau}\sum_{i\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}\times$$

$$E^{\mathbb{Q}_M}\left[H\left(S_t \exp[\{r - \frac{\sigma^2}{2} + (\frac{i\delta^2}{2\tau} - \frac{i\delta^2}{2\tau}) - \lambda(e^{\mu+\delta^2/2} - 1)\}\tau + i\mu + \sqrt{\sigma^2 + \frac{i\delta^2}{\tau}}B_\tau^{\mathbb{Q}_M}]\right)\right]$$

$$V^{Merton}(t, S_t) = e^{-r\tau}\sum_{i\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}\times$$

$$E^{\mathbb{Q}_M}\left[H\left(S_t \exp[\{r-\frac{1}{2}(\sigma^2+\frac{i\delta^2}{\tau})+\frac{i\delta^2}{2\tau}-\lambda(e^{\mu+\delta^2/2}-1)\}\tau+i\mu+\sqrt{\sigma^2+\frac{i\delta^2}{\tau}}B_\tau^{\mathbb{Q}_M}]\right)\right]$$

Set $\sigma_i^2=\sigma^2+\dfrac{i\delta^2}{\tau}$ and rearrange:

$$\begin{aligned}
&V^{Merton}(t,S_t)\\
&=e^{-r\tau}\sum_{i\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}E^{\mathbb{Q}_M}\left[H\left(S_t\exp[\{r-\frac{1}{2}\sigma_i^2+\frac{i\delta^2}{2\tau}-\lambda(e^{\mu+\delta^2/2}-1)\}\tau+i\mu+\sigma_i B_\tau^{\mathbb{Q}_M}]\right)\right]\\
&=e^{-r\tau}\sum_{i\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}E^{\mathbb{Q}_M}\left[H\left(S_t\exp\{i\mu+\frac{i\delta^2}{2}-\lambda(e^{\mu+\delta^2/2}-1)\tau\}\exp\{(r-\frac{1}{2}\sigma_i^2)\tau+\sigma_i B_\tau^{\mathbb{Q}_M}\}\right)\right].
\end{aligned}$$

Black-Scholes price can be expressed as:

$$V^{BS}(\tau=T-t,S_t;\sigma)=e^{-r\tau}E^{\mathbb{Q}_{BS}}[H\{S_t\exp(r-\frac{1}{2}\sigma^2)\tau+\sigma B_\tau^{\mathbb{Q}_{BS}}\}].$$

Finally, MJD pricing formula can be obtained as a weighted average of Black-Scholes price conditioned on the number of jumps i:

$$\begin{aligned}
&V^{Merton}(t,S_t)\\
&=\sum_{i\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}V^{BS}(\tau,S_i\equiv S_t\exp\{i\mu+\frac{i\delta^2}{2}-\lambda(e^{\mu+\delta^2/2}-1)\tau\};\sigma_i\equiv\sqrt{\sigma^2+\frac{i\delta^2}{\tau}}).
\end{aligned}\qquad(27)$$

Alternatively:

$$\begin{aligned}
&V^{Merton}(t,S_t)\\
&=e^{-r\tau}\sum_{i\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}E^{\mathbb{Q}_M}\left[H\left\{S_t\exp\{\left(r-\lambda(e^{\mu+\delta^2/2}-1)+\frac{i\mu+i\delta^2/2}{\tau}-\frac{1}{2}\sigma_i^2\right)\tau+\sigma_i B_\tau^{\mathbb{Q}_M}\}\right\}\right]\\
&=\sum_{i\geq 0}\frac{e^{-\bar{\lambda}\tau}(\bar{\lambda}\tau)^i}{i!}V^{BS}(\tau,S_t;\sigma_i\equiv\sqrt{\sigma^2+\frac{i\delta^2}{\tau}},r_i\equiv r-\lambda(e^{\mu+\delta^2/2}-1)+\frac{i\mu+i\delta^2/2}{\tau}).
\end{aligned}\qquad(28)$$

where $\bar{\lambda}=\lambda(1+k)=\lambda e^{\mu+\frac{1}{2}\delta^2}$. As you might notice, this is the same result as the option pricing formula derived from solving a PDE by forming a risk-free portfolio in equation (22). PDE approach and Martingale approach are different approaches but they are related and give the same result.

[7] Option Pricing Example of Merton Jump-Diffusion Model

In this section the equation (22) is used to price hypothetical plain vanilla options: current stock price $S_t = 50$, risk-free interest rate $r = 0.05$, continuously compounded dividend yield $q = 0.02$, time to maturity $\tau = 0.25$ years.

We need to be careful about volatility σ. In the Black-Scholes case, the t-period standard deviation of log-return x_t is:

$$\text{Standard Deviation}_{\text{BS}}(x_t) = \sigma_{BS}\sqrt{t} \ . \tag{29}$$

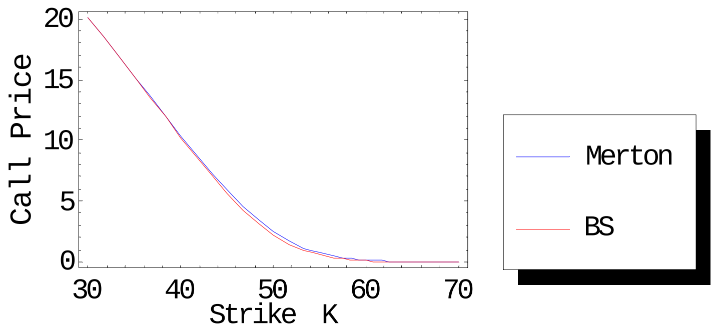
Equation (10) tells that the t-period standard deviation of log-return x_t in the Merton model is given as:

$$\text{Standard Deviation}_{\text{Merton}}(x_t) = \sqrt{(\sigma_{Merton}{}^2 + \lambda\delta^2 + \lambda\mu^2)t} \ . \tag{30}$$

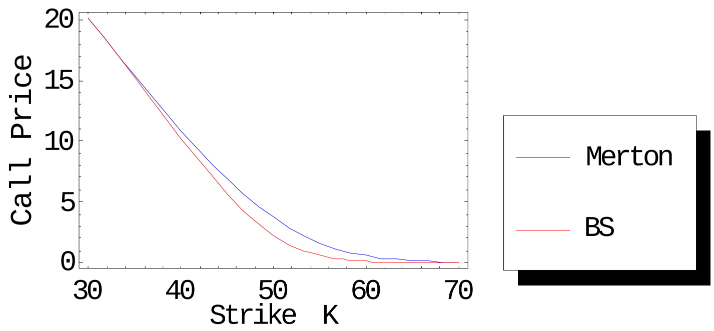
This means that if we set $\sigma_{BS} = \sigma_{Merton}$, MJD prices are always greater (or equal to) than Black-Scholes prices because of the extra source of volatility λ (intensity), μ (mean log-return jump size), δ (standard deviation of log-return jump) (i.e. larger volatility is translated to larger option price):

$$\begin{aligned} \text{Standard Deviation}_{\text{BS}}(x_t) &\leq \text{Standard Deviation}_{\text{Merton}}(x_t) \\ \sigma_{BS}\sqrt{t} &\leq \sqrt{(\sigma_{Merton}{}^2 + \lambda\delta^2 + \lambda\mu^2)t} \ . \end{aligned}$$

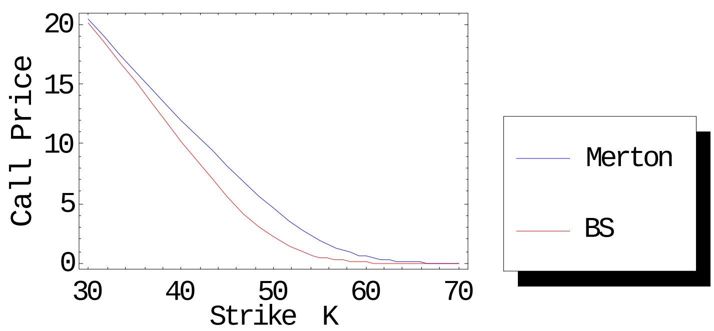
This very obvious point is illustrated in Figure 6 where diffusion volatility is set $\sigma_{BS} = \sigma_{Merton} = 0.2$. Note the followings: (1) In all four panels MJD price is always greater (or equal to) than BS price. (2) When Merton parameters (λ, μ, and δ) are small in Panel A, the difference between these two prices is small. (3) As intensity λ increases (i.e. increased expected number of jumps per unit of time), the t-period Merton standard deviation of log-return x_t increases (equation (30)) leading to the larger difference between Merton price and BS price as illustrated in Panel B. (4) As Merton mean log-return jump size μ increases, the t-period Merton standard deviation of log-return x_t increases (equation (30)) leading to the larger difference between Merton price and BS price as illustrated in Panel C. (5) As Merton standard deviation of log-return jump size δ increases, the t-period Merton standard deviation of log-return x_t increases (equation (30)) leading to the larger difference between Merton price and BS price as illustrated in Panel D.



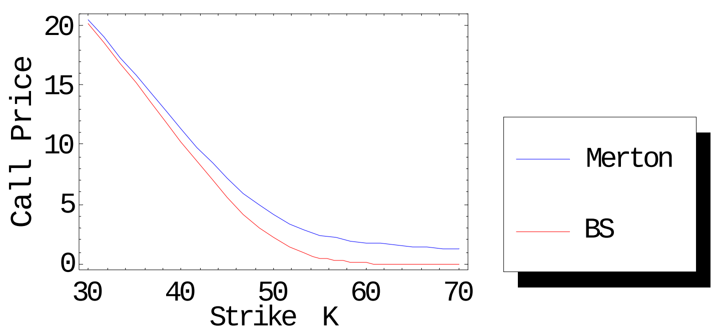
A) Merton parameters: $\lambda = 1$, $\mu = -0.1$, and $\delta = 0.1$.



B) Merton parameters: $\lambda = 5$, $\mu = -0.1$, and $\delta = 0.1$.



C) Merton parameters: $\lambda = 1$, $\mu = -0.5$, and $\delta = 0.1$.



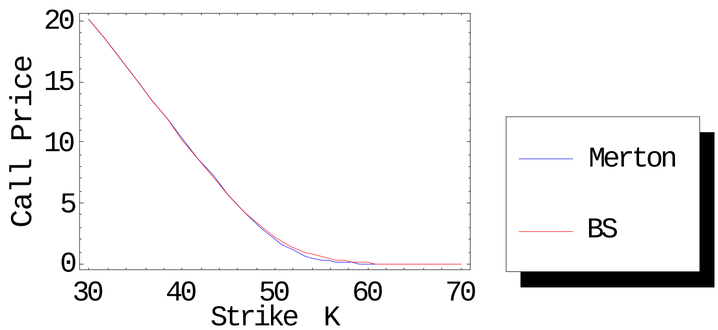
D) Merton parameters: $\lambda = 1$, $\mu = -0.1$, and $\delta = 0.5$.

Figure 6: MJD Call Price vs. BS Call Price When Diffusion Volatility σ is same. Parameters and variables used are $S_t = 50$, $r = 0.05$, $q = 0.02$, $\tau = 0.25$, and $\sigma_{BS} = \sigma_{Merton} = 0.2$.

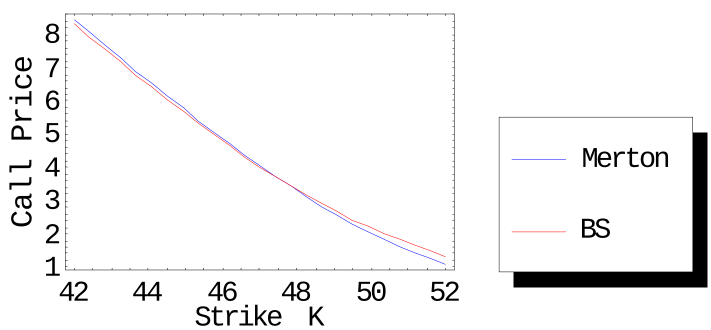
Next we consider a more delicate case where we restrict diffusion volatilities σ_{BS} and σ_{Merton} such that standard deviations of MJD and BS log-return densities are the same:

$$\text{Standard Deviation}_{\text{BS}}(x_t) = \text{Standard Deviation}_{\text{Merton}}(x_t),$$
$$\sigma_{BS}\sqrt{t} = \sqrt{(\sigma_{Merton}{}^2 + \lambda\delta^2 + \lambda\mu^2)t} .$$

Using the Merton parameters $\lambda = 1$, $\mu = -0.1$, and $\delta = 0.1$ and BS volatility $\sigma_{BS} = 0.2$, Merton diffusion volatility is calculated as $\sigma_{Merton} = 0.141421$. In this same standard deviation case, call price function is plotted in Figure 7 and put price function is plotted in Figure 8. It seems that MJD model overestimates in-the-money call and underestimates out-of-money call when compared to BS model. And MJD model overestimates out-of-money put and underestimates in-the-money put when compared to BS model.

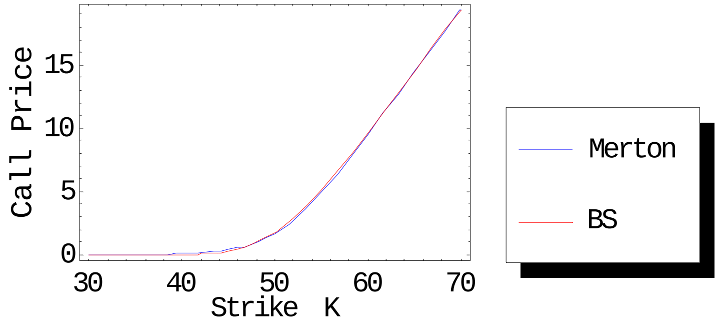


A) Range 30 to 70.

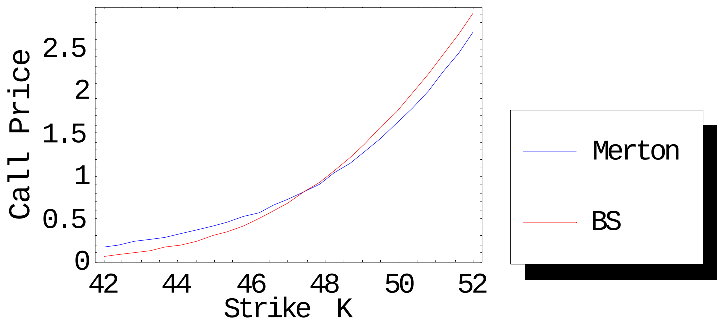


B) Range 42 to 52.

Figure 7: MJD Call Price vs. BS Call Price When Restricting Merton Diffusion Volatility σ_{Merton}**.** We set $\sigma_{BS} = 0.2$ and $\sigma_{Merton} = 0.141421$. Parameters and variables used are $S_t = 50$, $r = 0.05$, $q = 0.02$, $\tau = 0.25$.



A) Range 30 to 70.



B) Range 42 to 52.

Figure 8: MJD Put Price vs. BS Put Price When Restricting Merton Diffusion Volatility σ_{Merton}**.** We set $\sigma_{BS} = 0.2$ and $\sigma_{Merton} = 0.141421$. Parameters and variables used are $S_t = 50$, $r = 0.05$, $q = 0.02$, $\tau = 0.25$.

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