Dynamics of Risk-Neutral Densities Implied By Option Prices

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ABSTRACT

This paper investigates the dynamics of future price movements of the underlying financial assets contained in option prices. Two widely used but different methods for estimating option implied risk-neutral probability density functions (RNDs) of the underlying asset price on the expiration are employed. One is an implied volatility interpolation method by Shimko (1993), and the other is a mixture of two lognormals method by Bahra (1997). A daily time evolution of RNDs of S&P 500 index on December 19 2003 is estimated spanning December 23 2002 through December 18 2003 using the time series of S&P 500 futures option prices with December 2003 maturity. We find that the option implied dynamics of RNDs possesses well-documented features, negative skewness and excess kurtosis for asset prices relative to Black-Scholes lognormal dynamics of RNDs. This negative skewness and excess kurtosis generally increase as the maturity of the option approaches but within approximately 15 days to maturity excess kurtosis may decrease. A goodness of fit test reveals that the benchmark Black-Scholes lognormal RNDs do not come from the option implied RNDs for approximately one-third of the sample period at 5% significance level.

1. Introduction

Market participants' expectations of the future underlying asset price moves can be recovered from option prices because option prices depend on the market participants' expectations of the future underlying asset price moves. This option implied dynamics of risk-neutral probability density functions (RNDs) of future underlying asset price provide valuable information for risk managers in making investment decisions which is very different from the information provided by Benchmark Black-Scholes lognormal dynamics of RNDs. Previous studies[a] show that Black-Scholes assumption of constant volatility of the underling's return fails to capture the true dynamics of volatility and this leads to the failure of the Black-Scholes to capture the true dynamics of the expectation of future asset price move. For example, negatively sloped Black-Scholes implied volatilities across exercise prices on any trading day indicate the negatively skewed RND of underlying asset price. This means that option prices reveal that the likelihood of an extreme downward price move of the underlying is greater than that of an extreme upward move. Also the smile or sneer pattern of the Black-Scholes implied volatilities across exercise prices indicates that option prices reveal the likelihood of extreme move of the future underling's price is much greater than that allowed by the Black-Scholes. Thus obtaining the evolution in time of RNDs of financial asset prices from option prices can provide the dynamic behavior of market's assessment of risks.

Among numerous techniques to estimate the RND of underlying asset price on the option's maturity date using option prices, two popular but different classes of methods have been developed.

[a] For example, Rubinstein (1994), Dumas, Fleming, and Whaley (1998), and Cont and da Fonseca (2002).

The first class of method is based on the relationship that the RND of the underlying asset price at the option's maturity date can be obtained by twice differentiating the call price function with respect to the strike price discovered by Breeden and Litzenberger (1978). In order to take the second derivative of the call price function, a continuous call price function is necessary. Obviously, no options are traded at continuous strike price, rather they are traded at very limited number of strike prices. Thus, this method basically comes down to a method of interpolation and extrapolation. Bates (1991) employs a cubic spline (a peace-wise third order polynomials) to interpolate the observed call option prices subject to constraints. This approach requires a relatively large number of degrees of freedom because of the complex form of the call price function. Ait-Sahalia and Lo (1998) use a non-parametric kernel regression to estimate the call price function. This approach is not practical to implement because of its data-intensive nature. We need to make assumptions simply to reduce the dimensionality of the problem since this approach involves a large number of regressors. Instead of directly interpolating call option prices, Shimko (1993) first interpolates Black-Scholes implied volatilities by fitting a quadratic function across strike prices. Then this volatility smile is inverted to obtain a continuous call pricing function as a function of strike price through the Black-Scholes formula. Campa, Chang, and Reider (1998) interpolates Black-Scholes implied volatilities by fitting a cubic spline.

This type of semiparametric method imposes no specific dynamics on the underlying asset price and makes no assumptions about the parametric form of the RND. But the shortcoming is that the different methods of interpolation produce very different results in the form of RNDs.

The second class of method starts from imposing a particular parametric form on the RND and then recovers its parameters by the minimization between the observed option prices and model prices generated by the assumed parametric form.[b] Melick and Thomas (1997) use a mixture of three lognormals. Bahra (1997) uses a mixture of two lognormals. This is a more restrictive method than the first class.

Contrast to option implied methods to recover RND, the traditional methods start from imposing strong assumptions on the dynamics of underlying financial asset price. These strong assumptions enable RND to be estimated in closed form. Typically, geometric Brownian motion of the underlying asset price is assumed[c] which results in the benchmark Black-Scholes lognormal distribution of terminal asset price. This is the most restrictive case.

The goal of this paper is to assess the difference of the dynamics of RNDs between option implied methods and the benchmark Black-Scholes. None of the previous studies examined the difference in the dynamics of RNDs throughout the life of an option from the start till the end between the option implied and the traditional Black-Scholes.

A time series of S&P 500 futures option prices with December 2003 maturity traded on the Chicago Mercantile Exchange (CME) for the period of approximately one year between December 23 2002 and December 18 2003 is used. For each day in the sample, RND of underlying S&P 500 index on the option's expiration date December 19 2003 is estimated. The resulting dynamics of RNDs for the entire life of the option is plotted.

[b] Melick and Thomas (1997) points out that this approach is more general than specifying the dynamics of the underlying asset price. Because a particular dynamics gives a unique RND but a particular RND is consistent with many different dynamics of underlying asset price.
[c] Together with the constant volatility of asset price return and the constant risk-free interest rate.

We find that the option implied dynamics of RNDs is characterized by the well-documented features of negative skewness and excess kurtosis of the underlying asset price relative to Black-Scholes lognormal RNDs[d]. This feature becomes more pronounced as the maturity of the option approaches but may become less pronounced at very near maturity.

We examine the goodness of fit of the benchmark Black-Scholes lognormal dynamics to the option implied dynamics. Kolmogorov-Smirnov test (KS-test) concludes at the 5% significance level that the Black-Scholes RNDs do not come from the option-implied RNDs for approximately one-third of the sample period. The author attributes the observed pattern of the increasing deviation between the option-implied RNDs and the Black-Scholes RNDs when the maturity of the option is far and near to the use of volatility surface by the option-implied RNDs. This highlights the shortcoming of the Black-Scholes assumption of the constant volatility.

The remainder of the paper is organized as follows. Section 2 describes the methodology for estimating RNDs of financial asset price. Section 3 explains the data set of S&P 500 futures option prices with December 2003 maturity obtained from CME and presents the estimated dynamics of RNDs. We document the term structures of mean, standard deviation, skewness, and kurtosis for each of three different methods. Section 4 examines the goodness of fit of the Black-Scholes lognormal dynamics to the option implied dynamics of RNDs. Section 5 concludes.

We should be aware of the fact that the estimated RND is a risk-neutral distribution, not the actual distribution throughout this literature.

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[d] Rubinstein (1994) and Dumas, Fleming, and Whaley (1998).

2. Option Implied Method and Traditional Method to Estimate Risk-Neutral Density (RND) of Financial Asset Price

2.1 Volatility Interpolation (VI) Method of Shimko (1993)

Under Black-Scholes assumptions, the price of European call and put options at date t maturing at date $T \equiv t+\tau$, written on a stock with the price S_t at date t, and the strike price K can be written as the present value of the expected future payoffs following Cox, Ross, Rubinstein (1979):

$$\text{call}\left(K, \tau = T - t\right) = e^{-r\tau} \int_K^\infty \left(S_T - K\right) f\left(S_T\right) dS_T \tag{1}$$

$$\text{put}\left(K, \tau = T - t\right) = e^{-r\tau} \int_0^K \left(K - S_T\right) f\left(S_T\right) dS_T \tag{2}$$

where the present value is calculated with respect to the risk-free interest rate r. The expectation is calculated with respect to $f\left(S_T\right)$ which is the risk-neutral density (RND) of the stock at date T.

The explicit expression of RND can be obtained from option prices following Breeden and Litzenberger (1978). Rearrange the equation (1) as:

$$\text{call}\left(K, \tau\right) = e^{-r\tau} [\int_K^\infty S_T f\left(S_T\right) dS_T - K \int_K^\infty f\left(S_T\right) dS_T] \tag{3}$$

The partial derivative of call option price function (3) with respect to the strike price K becomes:

$$\frac{\partial \text{call}\left(K, \tau\right)}{\partial K} = e^{-r\tau} \left[-\int_K^\infty f\left(S_T\right) dS_T \right] = -e^{-r\tau} \int_K^\infty f\left(S_T\right) dS_T = -e^{-r\tau} \left[1 - F\left(S_T = K\right)\right] \tag{4}$$

where $F(S_T)$ is the risk-neutral cumulative density function of the stock price at date T.

$\int_K^{\infty} f(S_T)dS_T = 1 - F(S_T = K)$ is the probability that the stock price exceeds the strike price at the maturity (i.e. call option finishes in the money). The partial derivative of the equation (4) with respect to the strike price K gives the RND of stock price at date T:

$$\frac{\partial^2 \text{call}(K,\tau)}{\partial K^2} = e^{-r\tau} f(S_T = K) \tag{5}$$

Taking derivatives requires continuous call option pricing functions. Instead of directly interpolating call option prices,[e] Shimko (1993) first interpolates Black-Scholes implied volatilities using a quadratic function across strike prices.[f] Then this volatility smile is inverted to obtain a continuous call pricing function as a function of strike price through the Black-Scholes formula.[g]

A method of normalization is necessary for the recovered RND to behave nicely. This is for the RND to have an integral of one, for the tails of the RND to decline monotonically and decay quickly. For this purpose, lognormal probability density is grafted for strike prices outside the observed range by matching the density and cumulative density of the option implied density function with a lognormal probability density in both lower and upper tail.[h]

2.2 Mixture of Two Lognormals Method (2LN) of Bahra (1997)

This method begins by specifying the parametric form of RND directly as a mixture of two lognormal distributions. Bahra shows that the negative skewness and the excess

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[e] Directly interpolating call prices can produce inaccurate RND because small price errors can be transformed into large errors in the estimated RND, especially in the tails.
[f] Campa, Chang, and Reider (1998) interpolates the implied volatilities using cubic splines.
[g] Black-Scholes formula is merely used as a means of mapping option prices in terms of implied volatilities.
[h] This means that the implied volatilities are constant outside the range of traded strike prices.

kurtosis of the data are well captured by this parametric form and this parametric form ensures for the tails of the RND to decline monotonically and decay quickly. The RND is then estimated by minimizing the distance of the model-generated prices to the market option prices.

The RND is given a mixture of two-lognormal distributions:

$$f\left(S_T\right) = \pi L\left(\mu_1, \sigma_1; S_T\right) + \left(1-\pi\right) L\left(\mu_2, \sigma_2; S_T\right) \tag{6}$$

where $L\left(\mu_i, \sigma_i; S_T\right)$ is the probability density function of lognormal distribution meaning that whose logarithm is normal with mean μ_i and standard deviation σ_i . π is the weight assigned to each lognormal distributions. The mean and standard deviation parameters for each lognormal distribution μ_i and σ_i together with the weight assigned to each π determines the overall shape of the RND.

By substituting (6) into (1) and (2), now the call and put option pricing functions (1) and (2) can be written as:

$$\text{call}\left(K, \tau = T - t\right) = e^{-r\tau} \int_K^{\infty} \left[\pi L\left(\mu_1, \sigma_1; S_T\right) + \left(1-\pi\right) L\left(\mu_2, \sigma_2; S_T\right)\right]\left(S_T - K\right) dS_T \tag{7}$$

$$\text{put}\left(K, \tau = T - t\right) = e^{-r\tau} \int_0^{K} \left[\pi L\left(\mu_1, \sigma_1; S_T\right) + \left(1-\pi\right) L\left(\mu_2, \sigma_2; S_T\right)\right]\left(K - S_T\right) dS_T \tag{8}$$

Next substitute the density functions of lognormal distributions:

$$L\left(\mu_1, \sigma_1; S_T\right) = \frac{1}{S_T \sigma_1 \sqrt{2\pi}} \exp[-\frac{\left(\ln S_T - \mu_1\right)^2}{2\sigma_1^{\ 2}}] \tag{9}$$

$$L\left(\mu_2, \sigma_2; S_T\right) = \frac{1}{S_T \sigma_2 \sqrt{2\pi}} \exp[-\frac{\left(\ln S_T - \mu_2\right)^2}{2\sigma_2^{\ 2}}] \tag{10}$$

Bahra (1997) demonstrated that using change of variables twice allows a transformation from lognormal distributions to normal distributions.[i] Equations (7) and (8) have closed form solutions:

$$\text{call}\left(K,\tau\right)=e^{-r\tau}\left\{\begin{array}{l}\pi\left[\exp[\mu_1+\frac{1}{2}\sigma_1{}^2]N\left(d_1\right)-N\left(d_2\right)\right]+\\ \left(1-\pi\right)\left[\exp[\mu_2+\frac{1}{2}\sigma_2{}^2]N\left(d_3\right)-KN\left(d_4\right)\right]\end{array}\right\} \tag{11}$$

$$\text{put}\left(K,\tau\right)=e^{-r\tau}\left\{\begin{array}{l}\pi\left[-\exp[\mu_1+\frac{1}{2}\sigma_1{}^2]N\left(-d_1\right)+KN\left(-d_2\right)\right]+\\ \left(1-\pi\right)\left[-\exp[\mu_1+\frac{1}{2}\sigma_1{}^2]N\left(-d_3\right)+KN\left(-d_4\right)\right]\end{array}\right\} \tag{12}$$

where $d_1=\dfrac{-\ln S_T+\mu_1+\sigma_1{}^2}{\sigma_1}$, $d_2=d_1-\sigma_1$, $d_3=\dfrac{-\ln S_T+\mu_2+\sigma_2{}^2}{\sigma_2}$, and $d_4=d_3-\sigma_2$.

The RND is then estimated by minimizing the sum of squared errors between the fitted option prices by the model and market option prices across all exercise prices. The minimization problem is:

$$\underset{\mu_1,\mu_2,\sigma_1,\sigma_2,\pi}{\text{Min}}\quad \sum_{i=1}^{n}\left[\text{call}\left(K_i,\tau\right)-\text{call}^{\text{market}}{}_i\right]^2+\sum_{i=1}^{n}\left[\text{put}\left(K_i,\tau\right)-\text{put}^{\text{market}}{}_i\right]^2+ \\ \left[\pi\exp[\mu_1+\frac{1}{2}\sigma_1{}^2]+\left(1-\pi\right)\exp[\mu_2+\frac{1}{2}\sigma_2{}^2]-e^{r\tau}S_t\right] \tag{13}$$

subject to $\sigma_1,\sigma_2>0$ and $0\le\pi\le1$, over the range of observed strike prices $K_1,K_2,.....,K_n$. The mean of the RND should equal the forward price of the stock $F_t=S_te^{r\tau}$.

2.3 Traditional Method

[i] See Mathematical Appendix.

Traditional method is presented using general notations first and then it is applied to this paper's purpose.

Suppose that the value of variable x follows the general stochastic differential equation

$$dx = A(x,t)dt + B(x,t)dw \tag{14}$$

where the drift coefficient $A(x,t)$ and the diffusion coefficient $B(x,t)$ are functions of the variable x and date t and dw is a Wiener process. Probabilistic behavior of the variable x is represented by its conditional probability $p(x_T, T | x_t, t)$ in which x_t is the current value of variable x and t is the current date while x_T being the value of the variable x at some future date T. Fokker-Planck (FP) equation [j] [k]

$$\frac{\partial p(x_T, T | x_t, t)}{T} = -\frac{\partial (A(x_T, T) p(x_T, T | x_t, t))}{\partial x_T} + \frac{1}{2}\frac{\partial^2 (B(x_T, T)^2 p(x_T, T | x_t, t))}{\partial x_T{}^2} \tag{15}$$

gives the evolution of the probability density function of the variable x in time. FP equation is an equation of motion for the distribution function of stochastic variable x. It is parabolic in the sense that it involves a second derivative with respect to one variable x_T, and a first derivative with respect to the other variable T.[l]

Solving this linear second-order partial differential equation of parabolic type requires an initial condition and boundary conditions.

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[j] Also known as forward Kolmogorov equation.
[k] See Mathematical Appendix for its derivation.
[l] Equations of this type are known as heat or diffusion equations.

Consider a futures contract on a stock. The relationship between the futures price and the spot price is[m]

$$F_t = S_t e^{(r-q)(T-t)} \tag{16}$$

where F_t is the futures price at date t, S_t is the spot price at t, q is the rate of dividend yield, and T is the maturity date of the futures contract. Assume that the risk-neutral process for the spot price is given by geometric Brownian motion:

$$dS_t = (r-q)S_t dt + \sigma S_t dw \tag{17}$$

Applying Ito's lemma produces the futures price dynamics:

$$dF_t = \sigma F_t dw \tag{18}$$

This means that in a risk-neutral world the futures price follows geometric Brownian motion with zero drift coefficient and diffusion coefficient of σF_t.

Then the Fokker-Planck equation which describes the dynamics of the RND becomes

$$\frac{\partial p(F_T, T \mid F_t, t)}{T} = \frac{1}{2}\frac{\partial^2(\sigma^2 {F_T}^2 p(F_T, T \mid F_t, t))}{\partial {F_T}^2} \tag{19}$$

Note that at the maturity of the futures the futures price is equal to the asset's spot price $F_T = S_T$.

Utilizing the initial condition,

$$p(F_T, t \mid F_t, t) = \delta(F_T - F_t) \tag{20}$$

the equation (19) can be solved as:

[m] Under the usual assumptions that the risk-free interest rate is constant and equal to r for all maturities and the forward price for a contract with a certain delivery date is equal to the futures price for a contract with the same delivery date.

$$p(F_T, T \mid F_t, t) = \frac{1}{\sigma F_T \sqrt{2\pi(T-t)}} \exp[-\frac{[\ln(F_t / F_T) - \frac{1}{2}\sigma^2 (T-t)]^2}{2\sigma^2 (T-t)}] \quad (21)$$

This is the benchmark Black-Scholes lognormal RND.

3. Estimating the Dynamics of RNDs from S&P 500 Futures Options Data

To compare difference in the dynamics of RNDs obtained from three different estimators, an application to the data set of futures option prices on S&P 500 index is presented.

3.1 Data Set

Our data consist of daily settlement prices of futures options on the S&P 500 index with December 2003 maturity obtained from Chicago Mercantile Exchange (CME) Daily Bulletin for the period December 23, 2002, through December 18, 2003. The time to maturity ranges from 0.9918 years (362 days) to 0.0027 years (1 day). Two filters are applied. Options with prices less than 0.125 are eliminated. In the money options are eliminated because they are illiquid. Thus all options used are out of money options: for moneyness = strike price / futures price greater than 1 call prices are used and for moneyness less than 1 put prices are used. LIBOR in US dollars with nearest maturity is used as an appropriate risk-free interest rate.

Since these are American-style options, we use Barone-Adesi and Whaley (1987) quadratic approximation method to adjust for the early exercise premium.[n]

3.2 Dynamics of RND Estimates

To implement volatility interpolation (VI) method Black-Scholes implied volatilities are calculated and interpolated by a quadratic function across the range of strike prices on

[n] See Mathematical Appendix for this procedure.

each day. The resulting implied volatility surface is shown in Figure 1 as a function of the moneyness = strike price / futures price of the option and time to maturity. Figure 1 illustrates the well-documented feature of the implied volatility's dependence on moneyness and time-to-maturity[o]. On any single trading day the implied volatility is not constant over a range of moneyness and the shape of implied volatility curve changes as the time to maturity changes. As the maturity of the option approaches, the curve becomes steeper. This implied volatility surface is inverted to continuous call option prices through Black-Scholes formula. Twice differentiating this continuous call price function yields the RND. The resulting dynamics of RNDs obtained from volatility interpolation method is depicted in Figure 2. It captures the evolution in time of RNDs of S&P 500 index on the option's expiration date December 19 2003. As expected, RND becomes less dispersed as the option's expiration date approaches since the outcome becomes more certain at near maturity.

Mixture of two lognormals method (2LN) recovers the option implied RNDs by minimizing the distance between the model prices calculated by equations (11) and (12) and the observed option prices according to the criterion function equation (13). The resulting dynamics of RNDs is illustrated in Figure 4.

Solution of the Fokker-Planck (FP) equation assuming the geometric Brownian motion on the underlying asset price dynamics yields the benchmark Black-Scholes lognormal RND of equation (21). The annualized historical volatility is calculated in a usual fashion:

$$\sigma = \sqrt{252} \times \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \overline{R})^2}$$

[o]Rubinstein (1994), Dumas, Fleming, and Whaley (1998), and Cont and da Fonseca (2002).

where $R_i = \ln(\frac{F_t}{F_{t-1}})$.

Using spot futures price F_t and the historical volatility $\sigma = 0.167615335$, the evolution of RND in time is depicted in Figure 6.

3.3 Instantaneous Profile and the Term Structure of Moments

Since it is very difficult to notice the difference among Figures 2, 4, and 6, the instantaneous profiles of the entire dynamics of RNDs are taken on the following four dates, March 18, June 18, September 18, and November 18 2003. These correspond to approximately nine-, six-, three-, and one-month to maturity. These are reported in Figures 8 through 11 together with their moments on Tables 1 through 4. These instantaneous profiles show that the option implied RNDs possess heavier lower tails than the Black-Scholes RNDs suggesting that the market assesses the greater likelihood of extreme downward move of underlying asset price than allowed by the Black-Scholes.

The term structures of mean, standard deviation, skewness, and kurtosis of the RNDs are plotted in Figure 12.[p] Note the several differences between the option implied term structure and the Black-Scholes term structure. Although all three methods have same means and similar patterns of standard deviations, their skewness and kurtosis are remarkably different. The option implied dynamics is characterized by the negative skewness and excess kurtosis for asset prices documented by Ait-Sahalia and Lo (1998) which generally increase as the maturity of the option approaches. At very near maturity of within 15 days to expiration, the volatility interpolation method produces decreasing excess kurtosis of the underlying asset price. In contrast, the kurtosis recovered from mixture of two lognormals method continues to increase. Obviously, benchmark Black-

[p] Moments are directly computed from the each RND.

Scholes lognormal dynamics of RNDs completely fails to capture these dynamic behaviors of skewness and kurtosis.

4. Goodness of Fit Test

This paper uses Kolmogorov-Smirnov test (KS-test) to examine the difference between the option implied dynamics of RNDs and the Black-Scholes dynamics of RNDs. KS-test is one of the best known and widely used goodness of fit test for continuous distributions to determine if the empirical distribution function $F_n'(x)$ obtained from a sample of observations $x_1, x_2, ..., x_n$ comes from the hypothesized known population distribution function $F(x)$. We hypothesize that the option implied RND describes the true population distribution $F(x)$ and test if Black-Scholes RND which is the empirical distribution $F_n'(x)$ calculated using the historical volatility from 249 observations of futures option price comes from the $F(x)$. The KS-test statistic D is the maximum absolute deviation between $F_n'(x)$ and $F(x)$ across the range of the stochastic variable x:

$$D = \underset{x}{Max}\left|F_n'(x) - F(x)\right|$$

If the test statistic D is greater than the critical value[q], the null hypothesis is rejected and we conclude that the empirical distribution does not come from the hypothesized population distribution.

Figure 13 reports the result of KS-test of the Black-Scholes RNDs to the option implied RNDs obtained from volatility interpolation method at 5% significance level. Null hypothesis is rejected on 80 trading days out of 249 (approximately 32%). Figure 14

[q] Critical value is calculated by $\frac{1.36}{\sqrt{n}}$ where n is the sample size.

reports the result of KS-test of the Black-Scholes RNDs to the option implied RNDs obtained from mixture of two lognormals approach and rejects the null on 92 trading days out of 249 (approximately 37%). Note that the difference becomes significant when the maturity of the option is far and near. In this paper, Black-Scholes dynamics of RNDs is obtained using the historical volatility for the sample period. On the other hand, option implied dynamics incorporate the volatility surface portrayed in Figure 1. When the maturity of the option is far (near), at the money implied volatility is sizably larger (smaller) than the historical volatility as reported in Figure 16. The time series of absolute deviation between the two is plotted in the Figure 17 which resembles Figures 13 and 14 in that when the maturity of the option is far and near the difference becomes large.

The similarity of the two option implied dynamics is reported in Figure 15. The difference is very stable at approximately 2% except for a very short period of time at near maturity.

In summary, the difference between the option implied dynamics and the Black-Scholes dynamics of RNDs is attributed to the difference between the use of constant volatility by the Black-Scholes and the use of volatility surface by the option implied. Volatility surface captures the dependence of volatility on moneyness and the evolution of this volatility curve as the time to maturity of the option changes.

5. Conclusion

This paper investigates the dynamics of risk-neutral probability density functions (RNDs) of S&P 500 index using a time series of S&P 500 futures options prices. Two different methods, volatility interpolation method and mixture of two lognormals method, for estimating RNDs from option prices are used. The difference in the dynamic behavior of RNDs between the option implied method and the benchmark Black-Scholes lognormal method was then examined.

We find that the option implied term structure of RNDs is characterized by the persistent negative skewness and excess kurtosis which become more pronounced as the maturity of the option approaches but may become less pronounced at very near maturity of within 15 days to expiration. In contrast, the benchmark Black-Scholes lognormal dynamics yields completely different dynamics of skewness and kurtosis, persistent positive skewness and time-decreasing kurtosis.

Next we performed a goodness of fit test of the benchmark Black-Scholes lognormal RNDs to the option implied RNDs. The results from Kolmogorov-Smirnov test suggest that at 5% significance level Black-Scholes lognormal RNDs do not come from option-implied RNDs on approximately one-third of the life of the option. The author attributes the observation that the difference between Black-Scholes RNDs and the option implied RNDs becomes significant when the maturity of the option is far and near to the use of volatility surface by the option implied RNDs.

Overall, our results indicate that option prices do not support the Black-Scholes world of constant volatility and the lognormal distribution of asset prices at option's expiration. The use of information provided by volatility surface is a key element in option pricing.

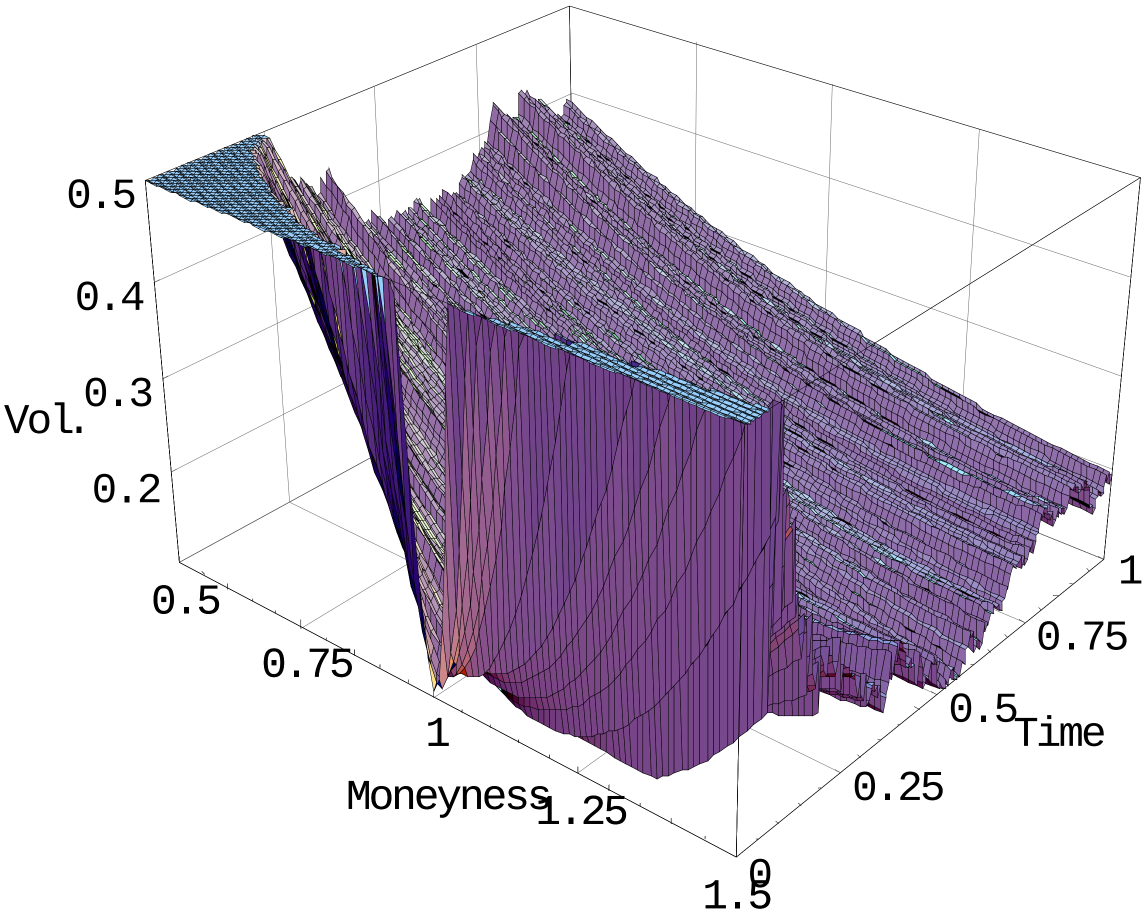


Figure 1. Implied Volatility Surface. Plot of the dynamics of implied volatility curves of futures options on S&P 500 with December 2003 maturity as a function of time to maturity in years and moneyness = Strike/Futures for the period between December 23 2002 and December 18 2003. Quadratic function is used as a method of interpolation.

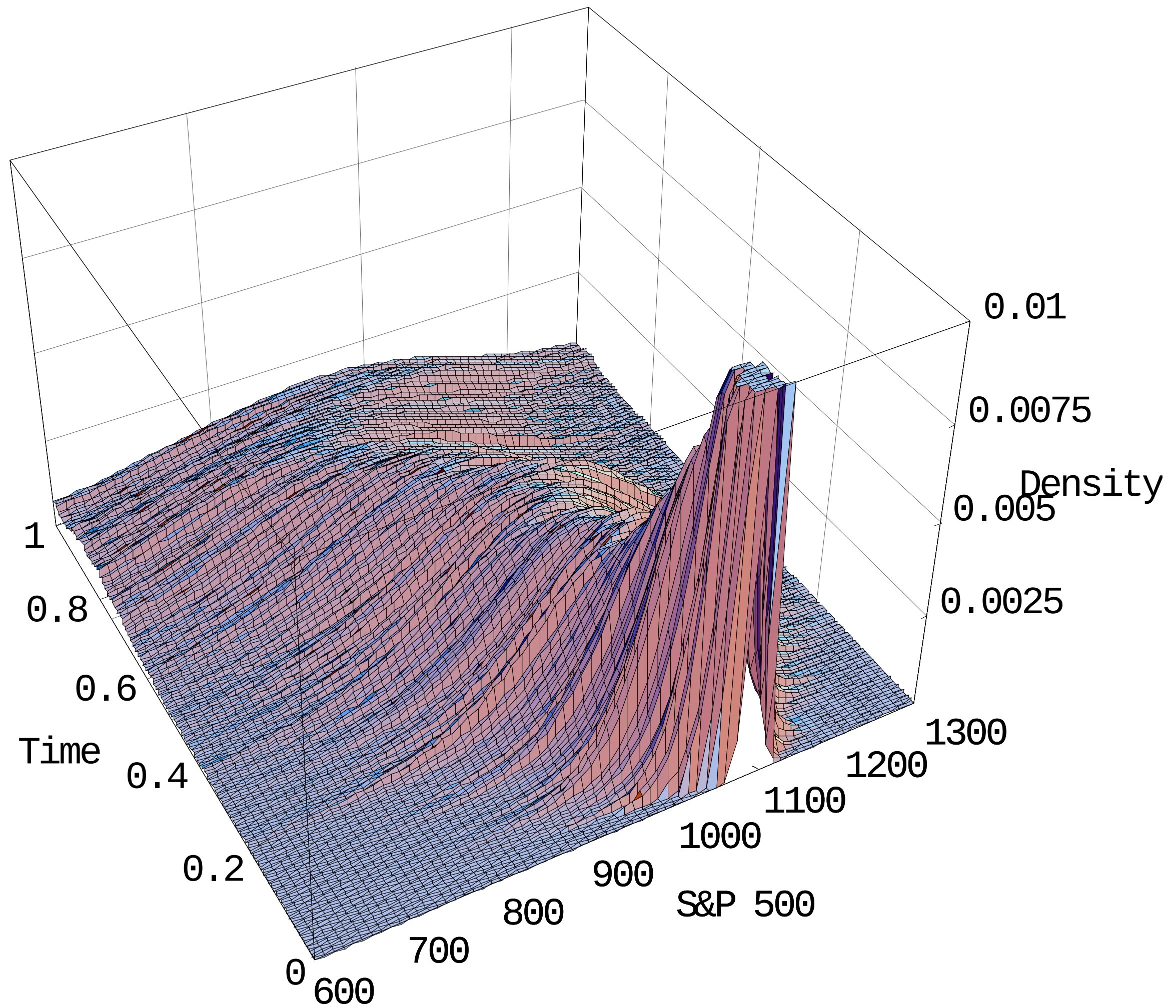


Figure 2: Dynamics of Risk-Neutral Densities Obtained from Volatility Interpolation Method. Plot of the dynamics of risk-neutral densities of futures options on S&P 500 with December 2003 maturity obtained from volatility interpolation method for the period between December 23 2002 and December 18 2003. Time to maturity is measured in years. Range of the density plotted is between 0 and 0.01.

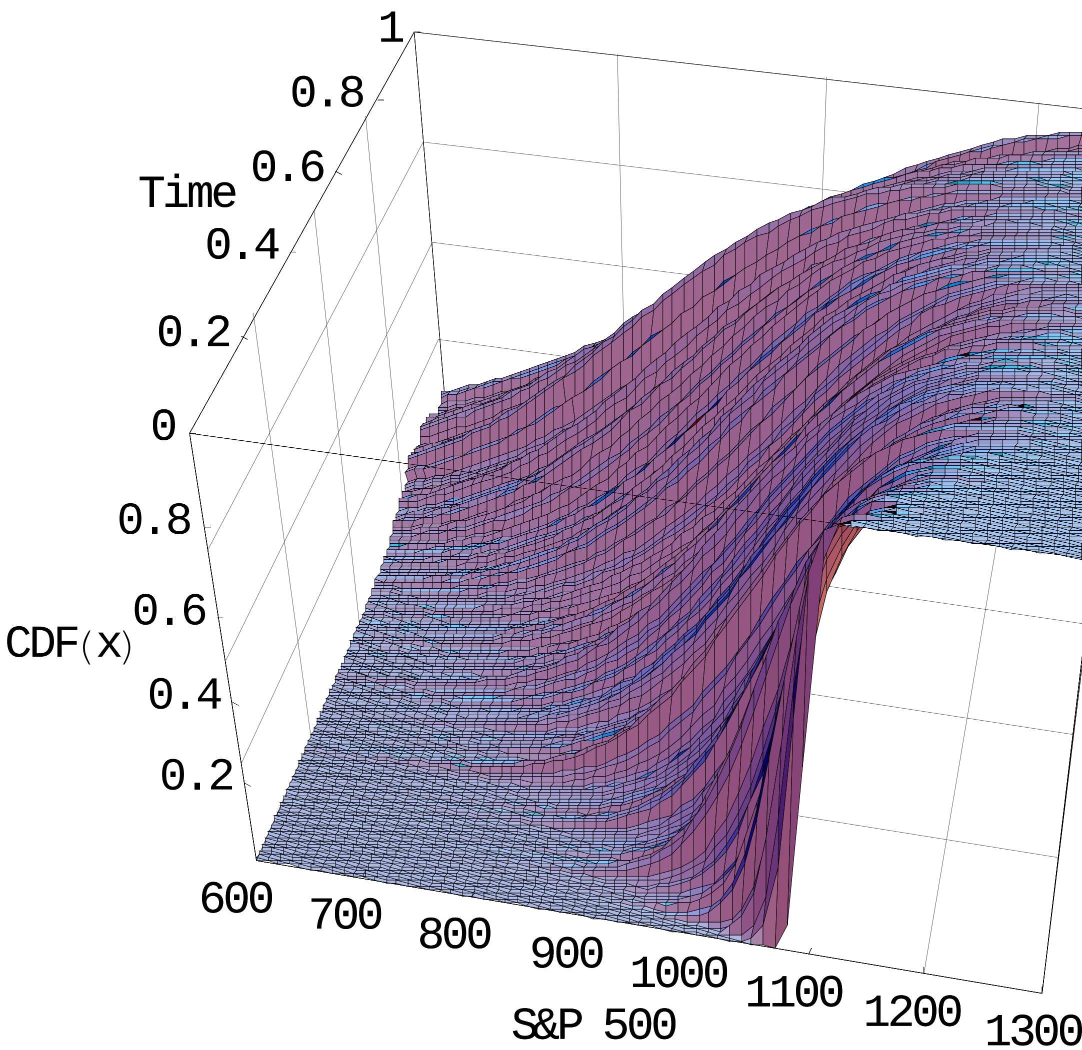


Figure 3: Dynamics of Cumulative Risk-Neutral Densities Obtained from Volatility Interpolation Method. Plot of the dynamics of cumulative risk-neutral densities of futures options on S&P 500 with December 2003 maturity obtained from volatility interpolation method for the period between December 23 2002 and December 18 2003. Time to maturity is measured in years.

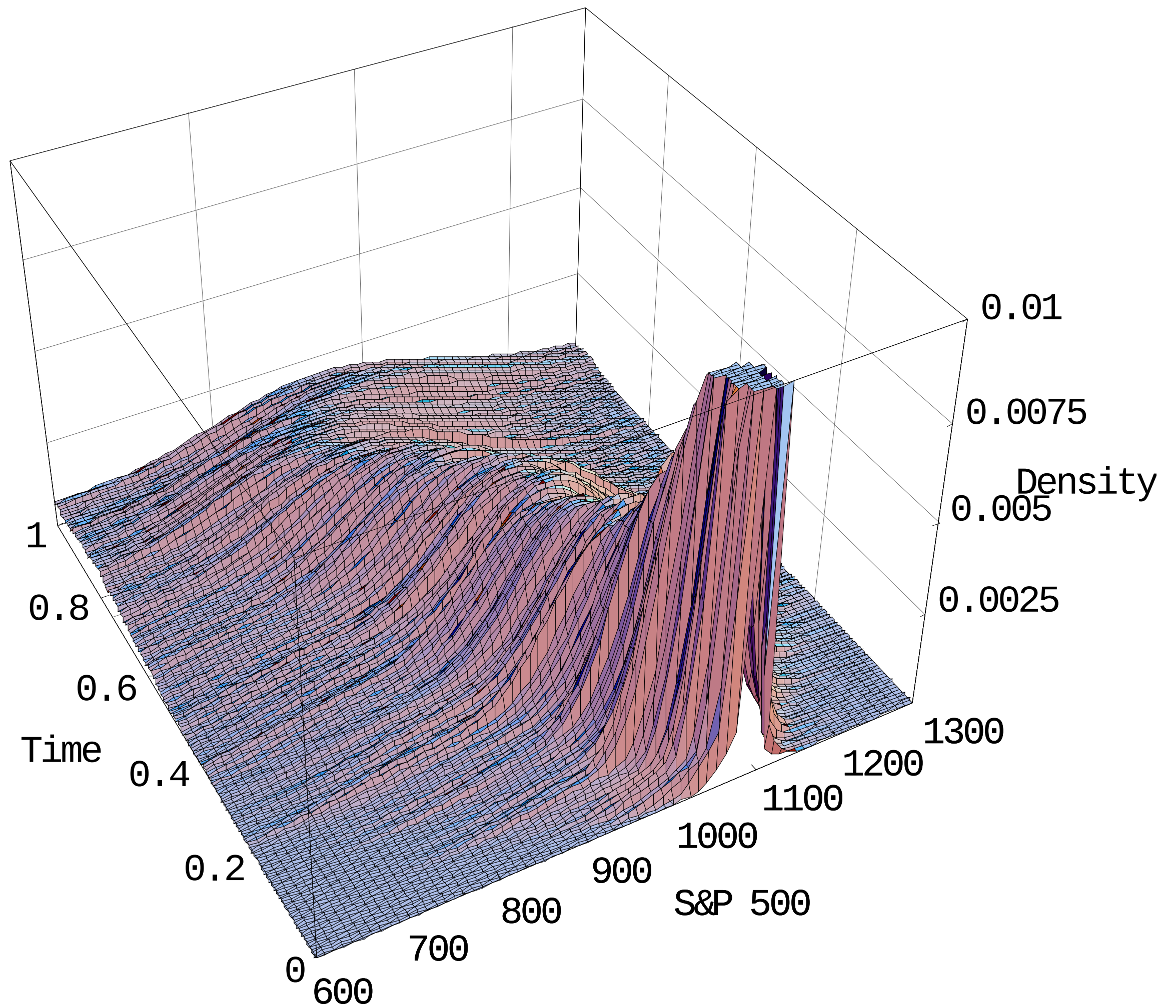


Figure 4: Dynamics of Risk-Neutral Densities Obtained from Mixture of Two Lognormals Method. Plot of the dynamics of risk-neutral densities of futures options on S&P 500 with December 2003 maturity obtained from mixture of two lognormals method for the period between December 23 2002 and December 18 2003. Time to maturity is measured in years. Range of the density plotted is between 0 and 0.01.

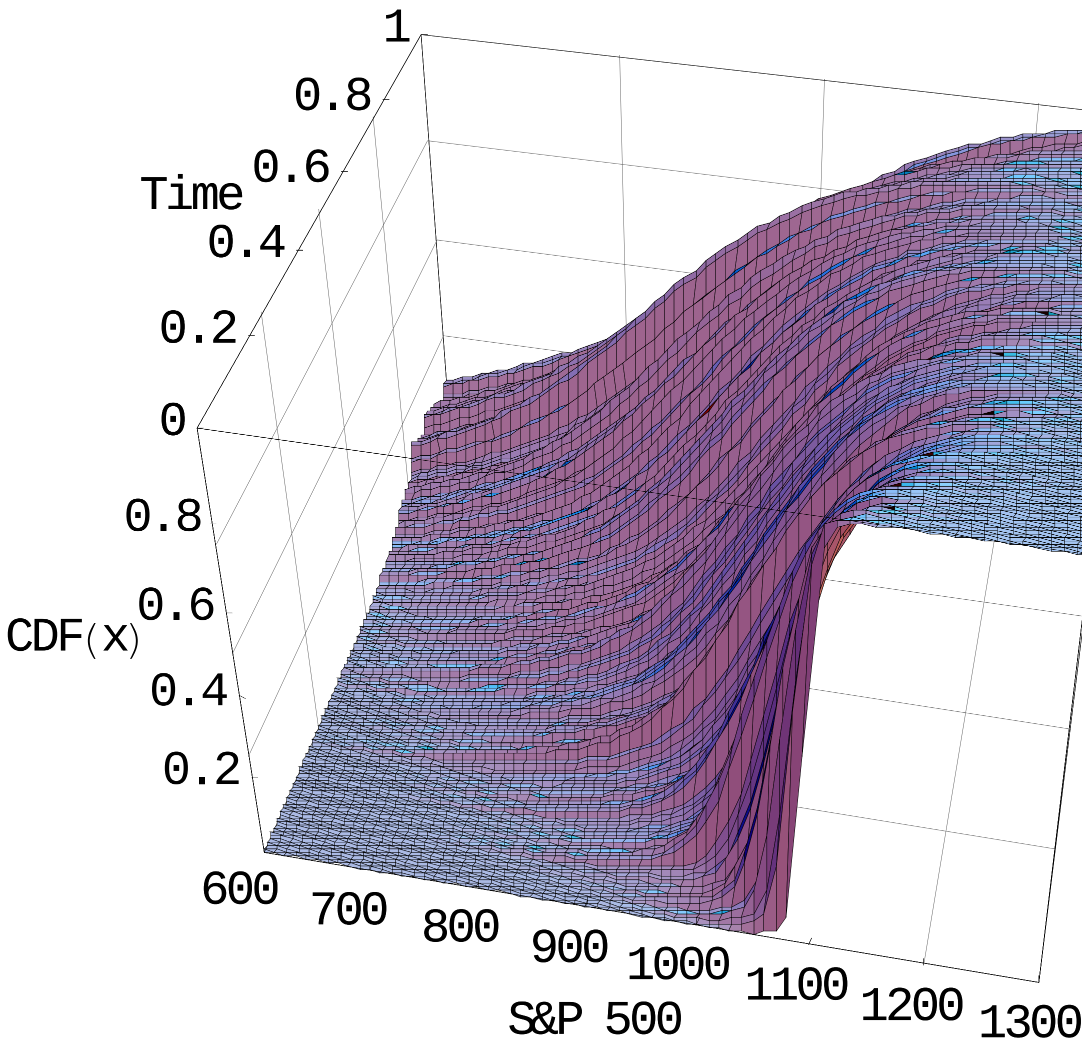


Figure 5: Dynamics of Cumulative Risk-Neutral Densities Obtained from Mixture of Two Lognormals Method. Plot of the dynamics of cumulative risk-neutral densities of futures options on S&P 500 with December 2003 maturity obtained from mixture of two lognormals method for the period between December 23 2002 and December 18 2003. Time to maturity is measured in years.

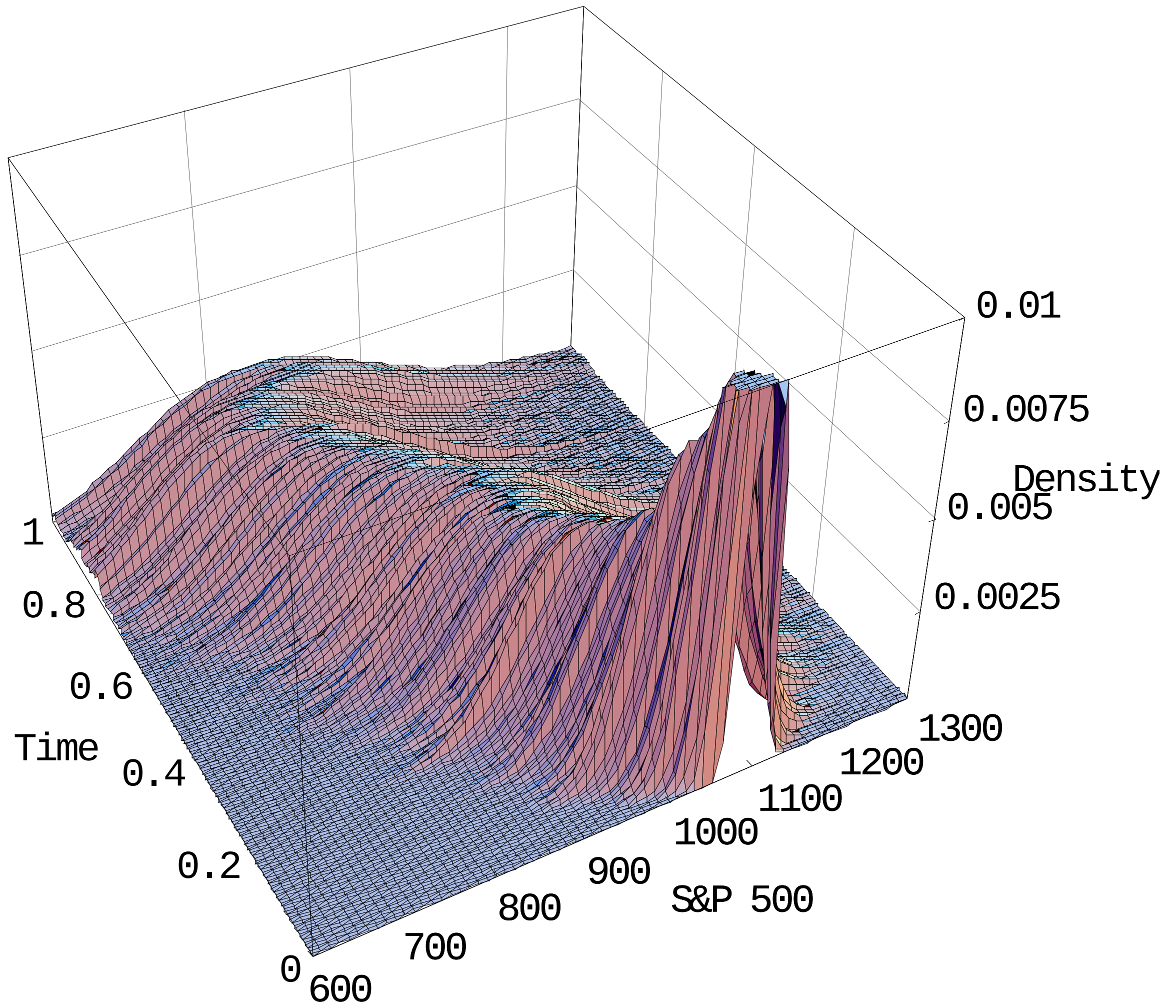


Figure 6: Dynamics of Risk-Neutral Densities Obtained from a Benchmark Black-Scholes Lognormal RND. Plot of the dynamics of risk-neutral densities of futures options on S&P 500 with December 2003 maturity obtained from a benchmark Black-Scholes lognormal RND for the period between December 23 2002 and December 18 2003. Time to maturity is measured in years. Range of the density plotted is between 0 and 0.01.

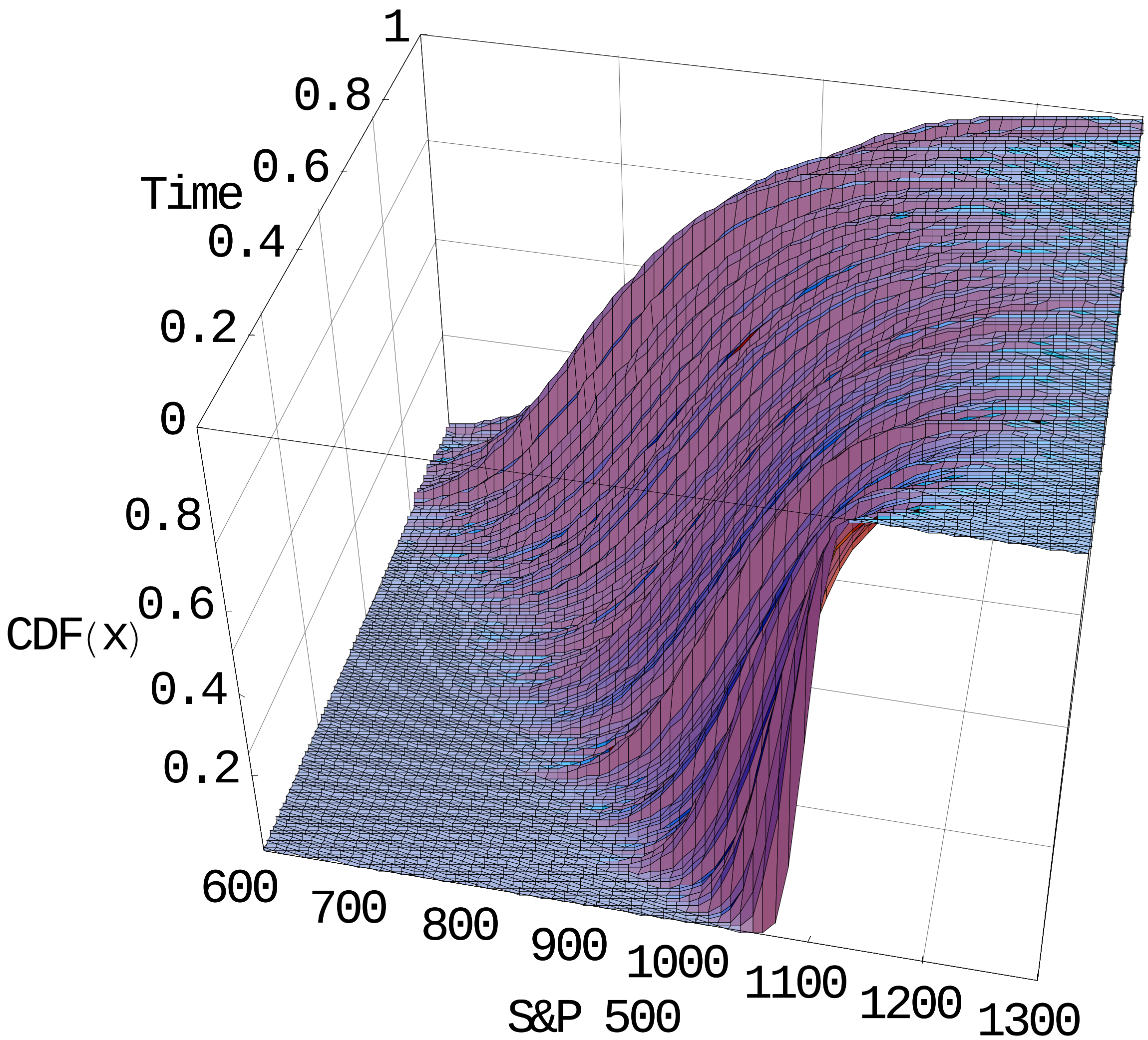


Figure 7: Dynamics of Cumulative Risk-Neutral Densities Obtained from a Benchmark Black-Scholes Lognormal RND. Plot of the dynamics of cumulative risk-neutral densities of futures options on S&P 500 with December 2003 maturity obtained from a benchmark Black-Scholes lognormal RND for the period between December 23 2002 and December 18 2003. Time to maturity is measured in years.

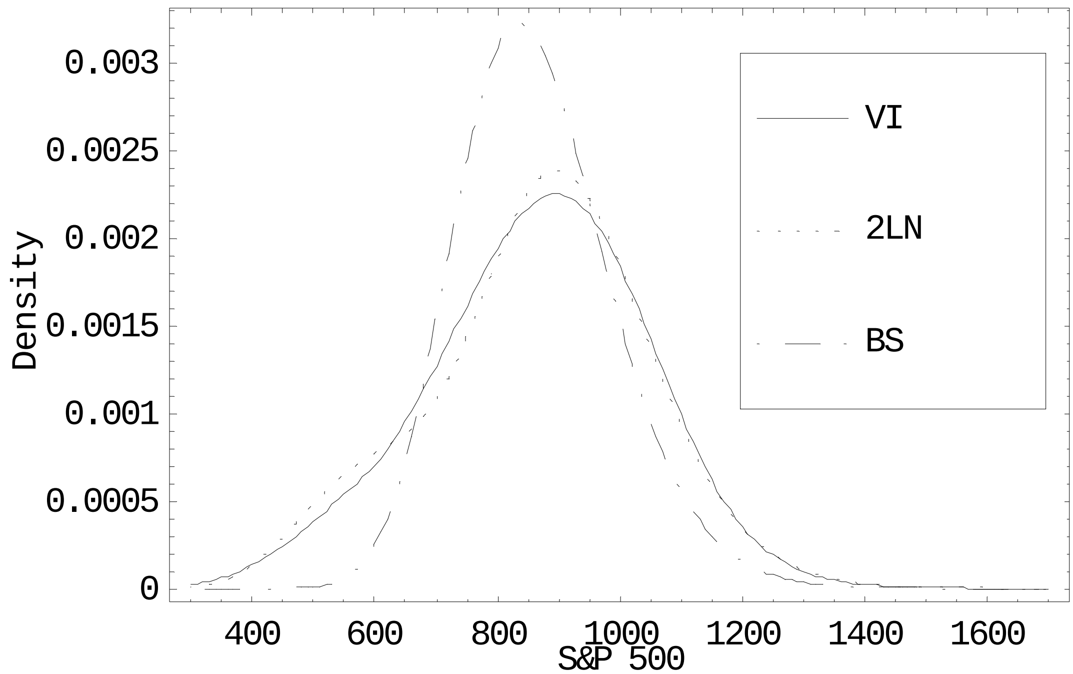


Figure 8: Risk-Neutral Density Estimates with 9-Month to Maturity. Plot of risk-neutral density estimates of futures options on S&P 500 with December 2003 maturity on March 18 2003 obtained from three different methods.

Table 1
Moments of RND Estimates with 9-Month to Maturity

Moments of RND recovered from volatility interpolation (VI), mixture of two lognormals (2LN), and the benchmark Black-Scholes lognormal (BS) methods. This table quantifies the difference of RND estimates among three methods with respect to their first four moments. The estimates are based on CME settlement prices of futures options on S&P 500 index with December 2003 maturity on March 18, 2003.

RND Estimator	Mean	Standard Deviation	Skewness	Kurtosis
VI	864.779	183.395	-0.1901	3.0095
2LN	863.474	185.57	-0.1834	2.9894
BS	863.4	127.000	0.4445	3.3533

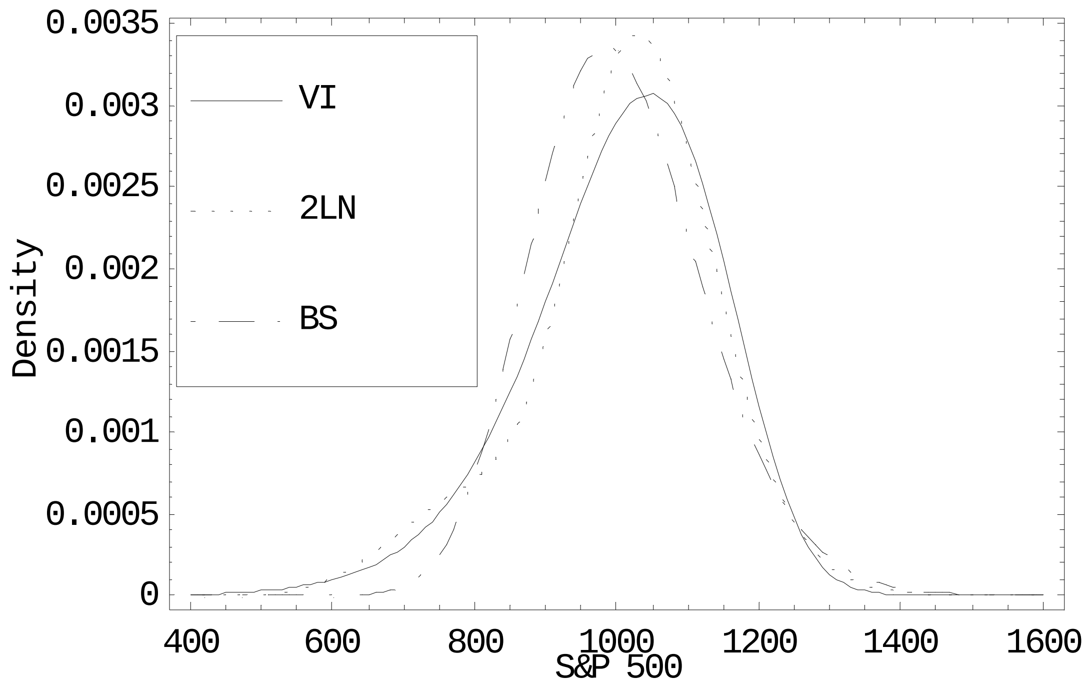


Figure 9: Risk-Neutral Density Estimates with 6-Month to Maturity. Plot of risk-neutral density estimates of futures options on S&P 500 with December 2003 maturity on June 18 2003 obtained from three different methods.

Table 2

Moments of RND Estimates with 6-Month to Maturity

Moments of RND recovered from volatility interpolation (VI), mixture of two lognormals (2LN), and the benchmark Black-Scholes lognormal (BS) methods. This table quantifies the difference of RND estimates among three methods with respect to their first four moments. The estimates are based on CME settlement prices of futures options on S&P 500 index with December 2003 maturity on June 18, 2003.

RND Estimator	Mean	Standard Deviation	Skewness	Kurtosis
VI	1006.465	137.243	-0.5682	3.5095
2LN	1006.305	136.252	-0.4972	3.5004
BS	1006.4	120.654	0.3614	3.2331

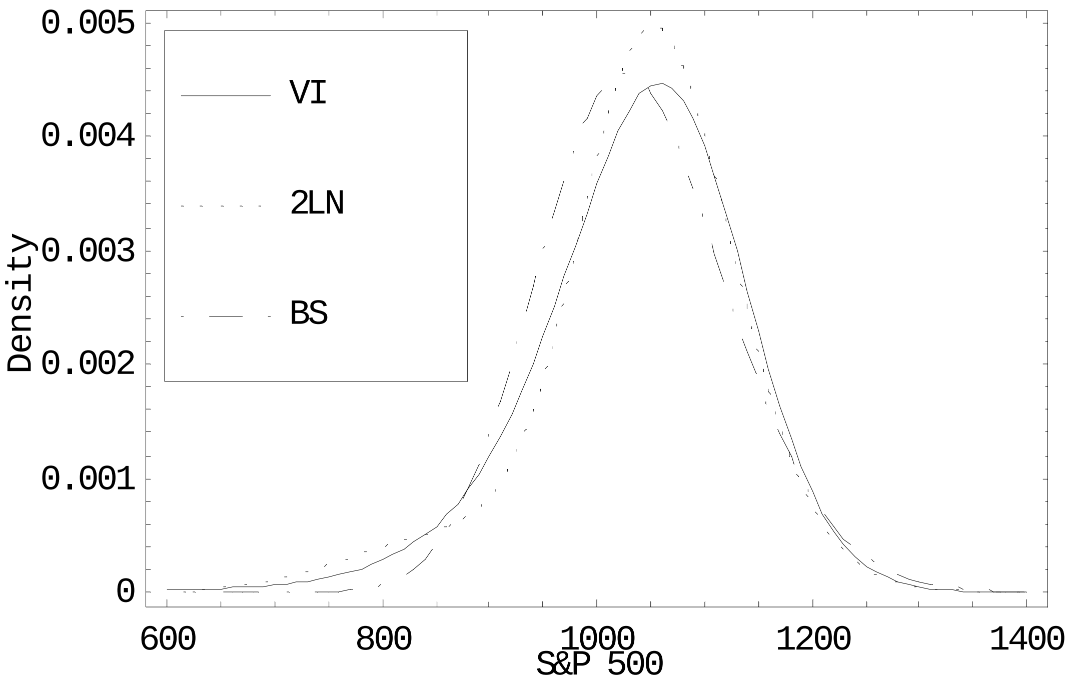


Figure 10: Risk-Neutral Density Estimates with 3-Month to Maturity. Plot of risk-neutral density estimates of futures options on S&P 500 with December 2003 maturity on September 18 2003 obtained from three different methods.

Table 3
Moments of RND Estimates with 3-Month to Maturity

Moments of RND recovered from volatility interpolation (VI), mixture of two lognormals (2LN), and the benchmark Black-Scholes lognormal (BS) methods. This table quantifies the difference of RND estimates among three methods with respect to their first four moments. The estimates are based on CME settlement prices of futures options on S&P 500 index with December 2003 maturity on September 18, 2003.

RND Estimator	Mean	Standard Deviation	Skewness	Kurtosis
VI	1037.345	100.405	-0.7738	5.2039
2LN	1037.014	98.8	-0.7555	4.4533
BS	1037.3	88.463	0.2565	3.1172

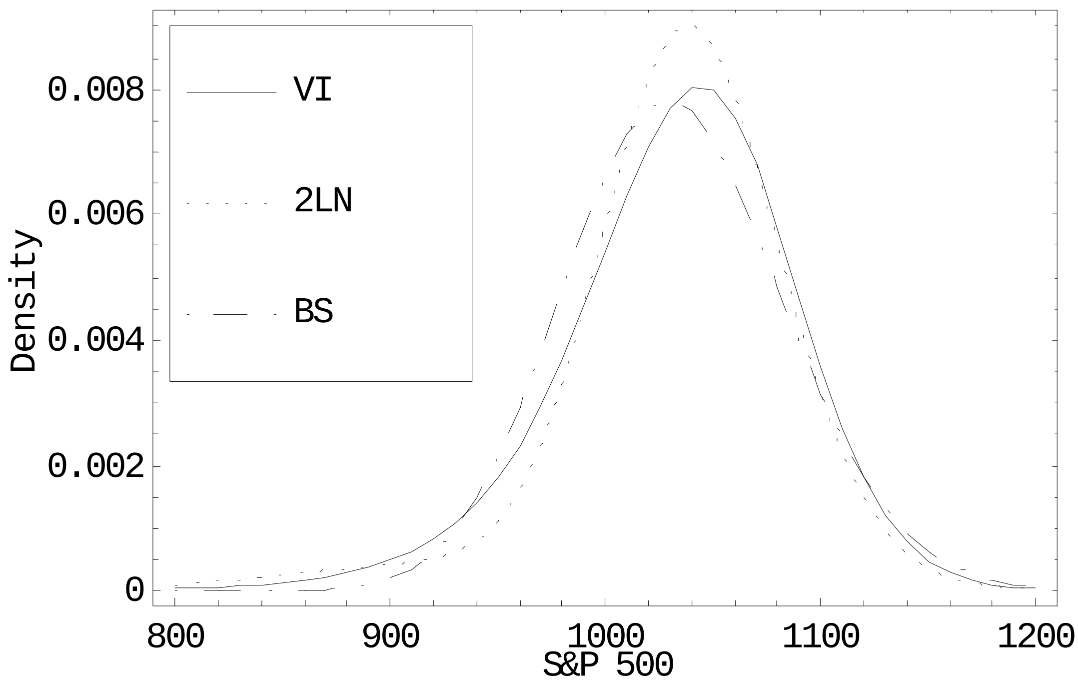
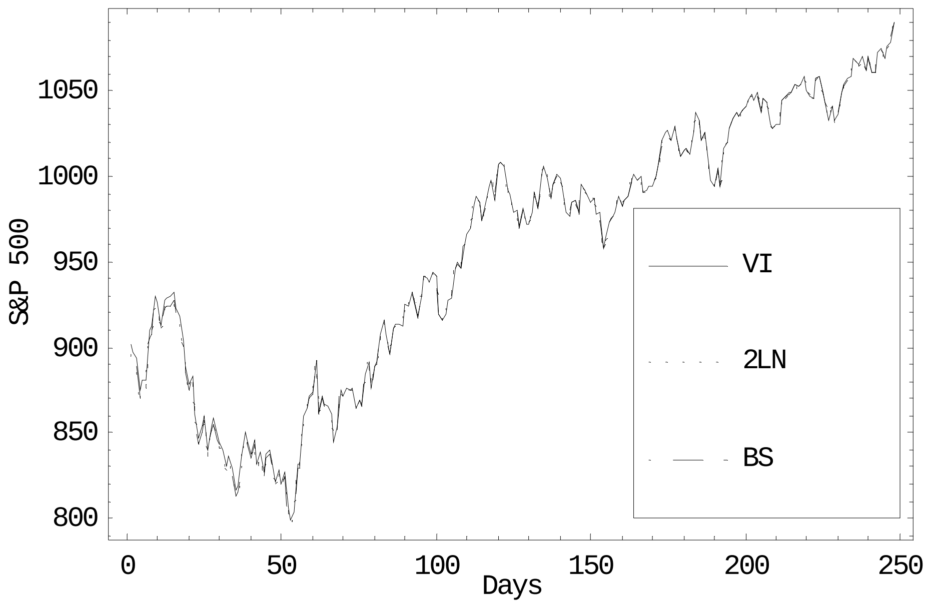


Figure 11: Risk-Neutral Density Estimates with 1-Month to Maturity. Plot of risk-neutral density estimates of futures options on S&P 500 with December 2003 maturity on November 18 2003 obtained from three different methods.

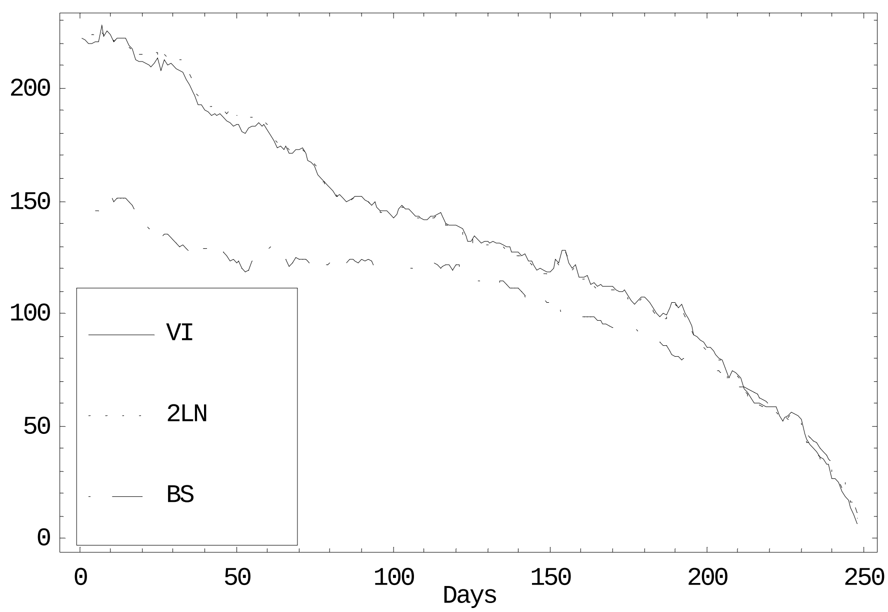
Table 4
Moments of RND Estimates with 1-Month to Maturity

Moments of RND recovered from volatility interpolation (VI), mixture of two lognormals (2LN), and the benchmark Black-Scholes lognormal (BS) methods. This table quantifies the difference of RND estimates among three methods with respect to their first four moments. The estimates are based on CME settlement prices of futures options on S&P 500 index with December 2003 maturity on November 18, 2003.

RND Estimator	Mean	Standard Deviation	Skewness	Kurtosis
VI	1033.045	56.118	-0.7328	5.0895
2LN	1032.409	55.536	-1.0367	6.4456
BS	1032.8	51.181	0.1488	3.0394

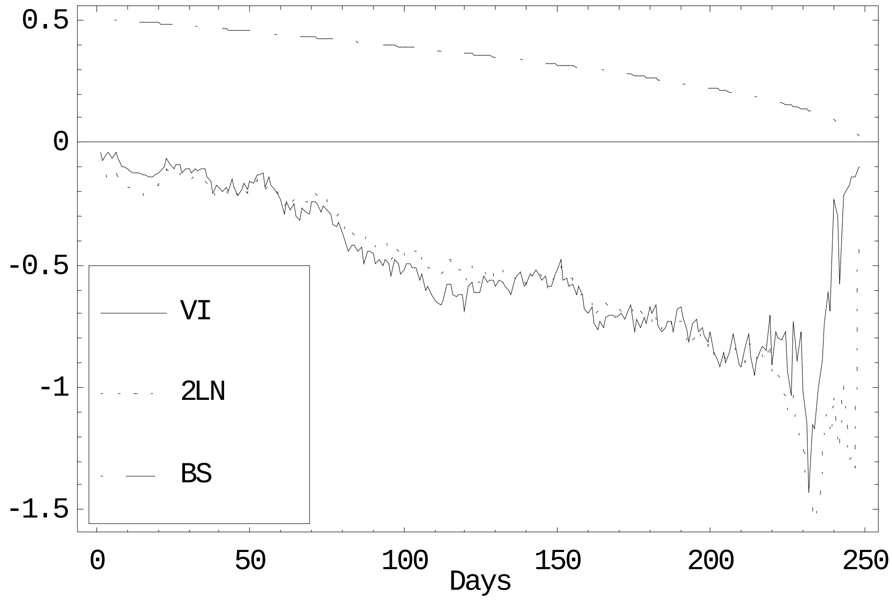


A. Mean

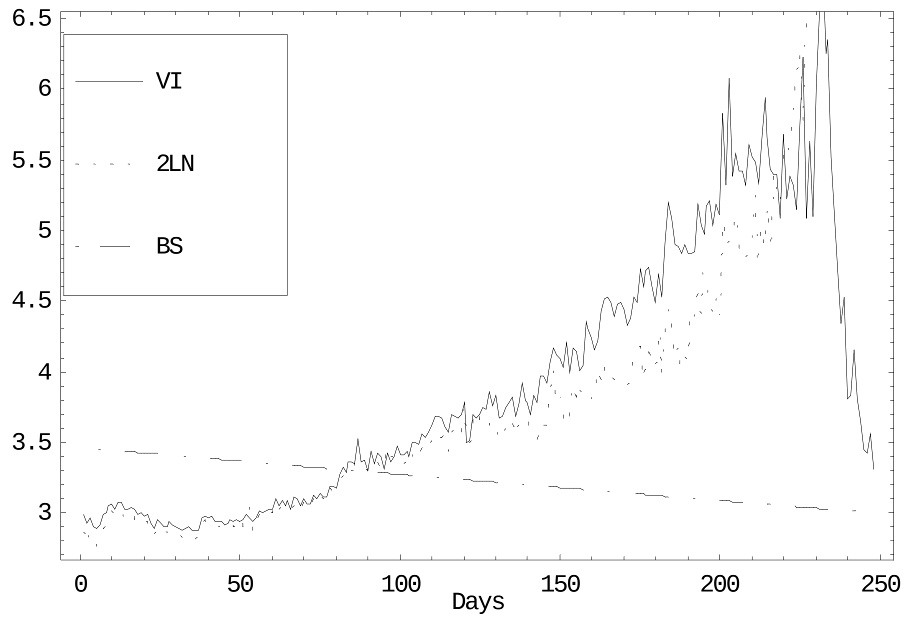


B. Standard Deviation

Figure 12: Term Structure of Moments of Risk-Neutral Density Estimates. Plot of the term structure of mean, standard deviation, skewness, and kurtosis of risk-neutral density estimates of futures options on S&P 500 with December 2003 maturity for the period between December 23 2002 and December 18 2003.

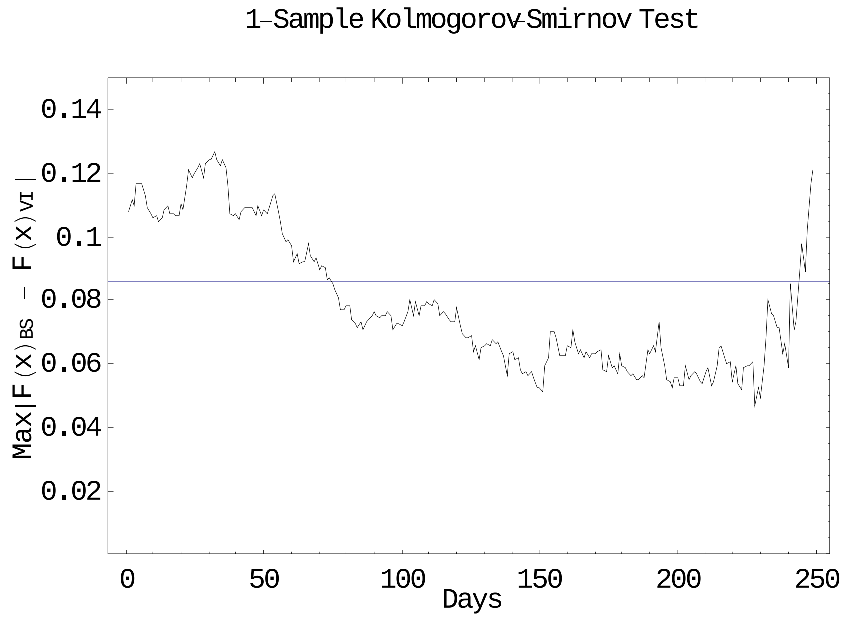


C. Skewness

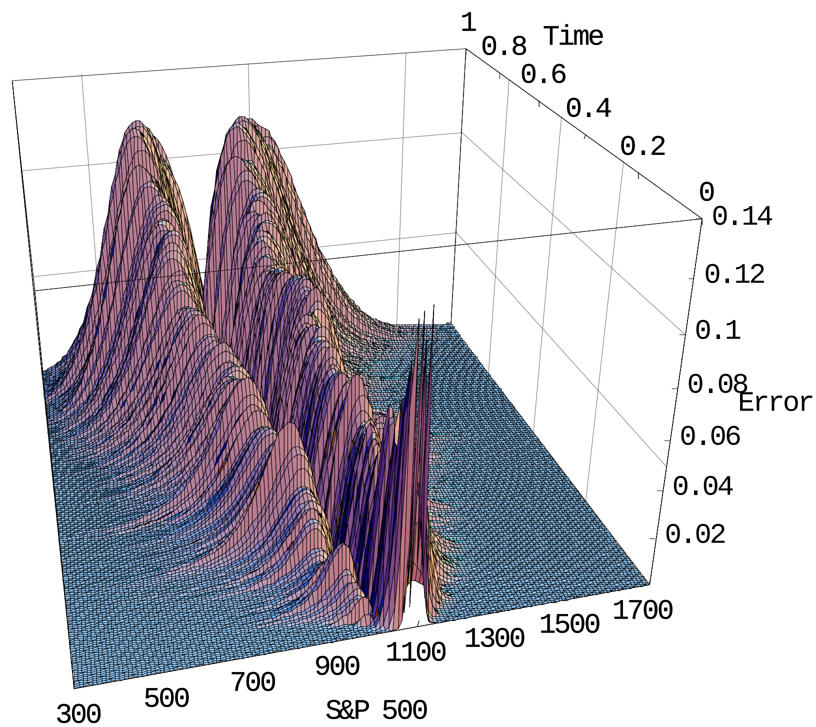


D. Kurtosis

Figure 12 (Continued): Term Structure of Moments of Risk-Neutral Density Estimates. Plot of the term structure of mean, standard deviation, skewness, and kurtosis of risk-neutral density estimates of futures options on S&P 500 with December 2003 maturity for the period between December 23 2002 and December 18 2003.

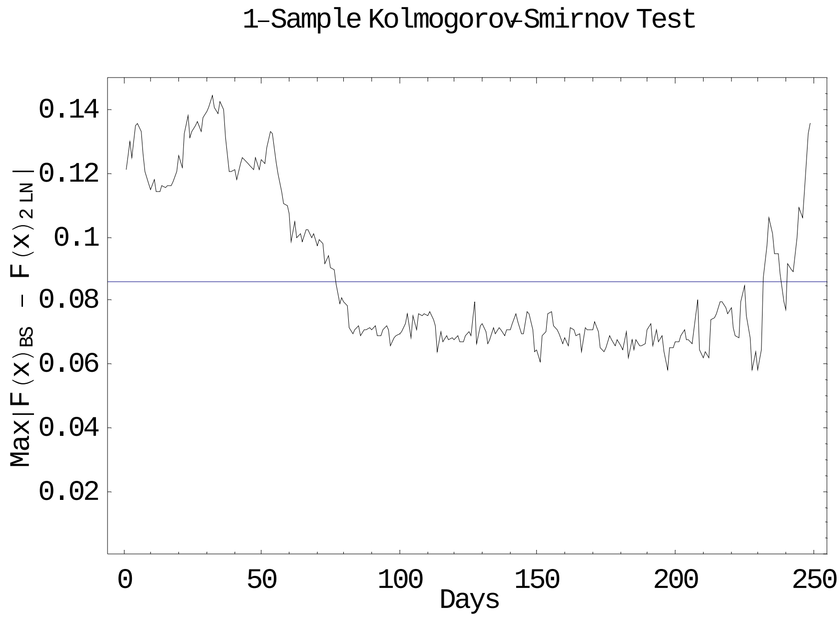


A. Plot of Test Statistic D Value $Max\left|F_{BS}(x) - F_{VI}(x)\right|$

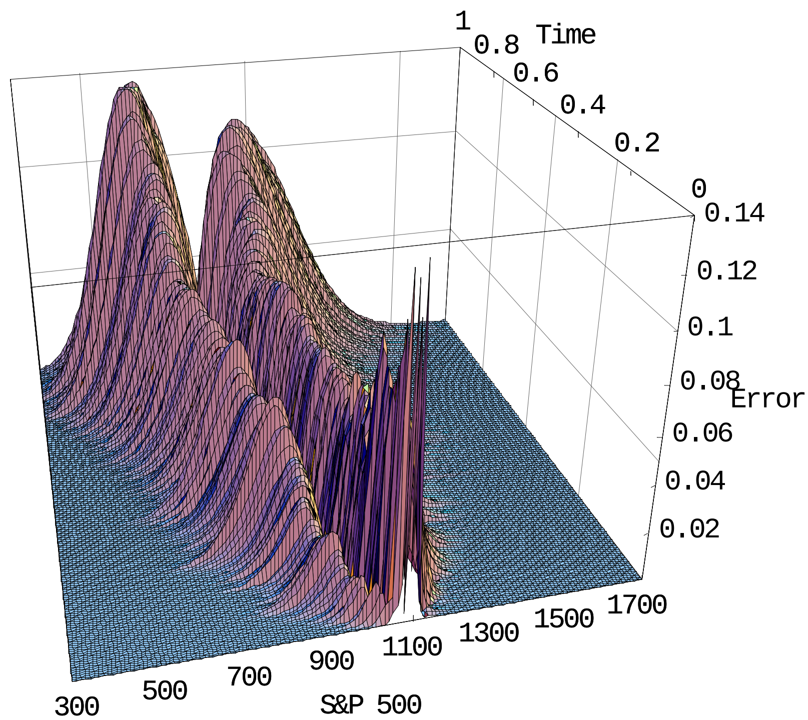


B. Plot of $\left|F_{BS}(x) - F_{VI}(x)\right|$

Figure 13: Kolmogorov-Smirnov Test of Risk-Neutral Density Estimates between Volatility Interpolation and Benchmark Black-Scholes Lognormal Methods. Plot of the largest absolute deviation of the cumulative distribution functions in A and the absolute deviation across index value in B obtained from volatility interpolation and Black-Scholes lognormal methods for the period between December 23 2002 and December 18 2003. Critical value is 0.0861865.

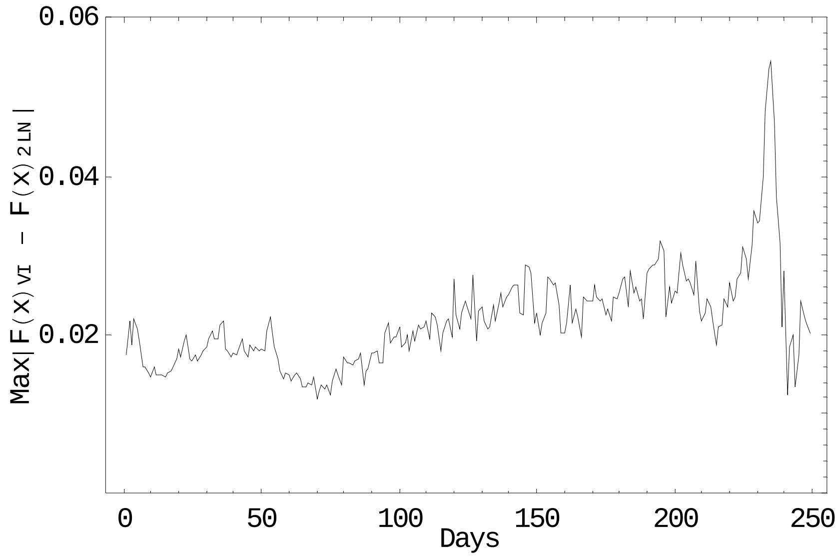


A. Plot of Test Statistic D Value $Max\left|F_{BS}(x)-F_{2LN}(x)\right|$

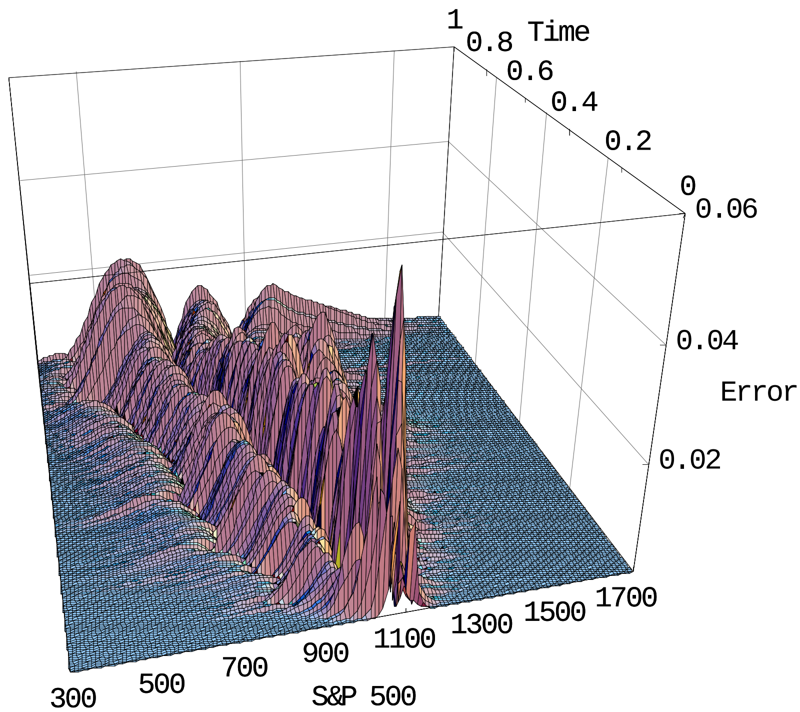


B. Plot of $\left|F_{BS}(x)-F_{2LN}(x)\right|$

Figure 14: Kolmogorov-Smirnov Test of Risk-Neutral Density Estimates between Mixture of Two Lognormals and Benchmark Black-Scholes Lognormal Methods. Plot of the largest absolute deviation of the cumulative distribution functions in A and the absolute deviation across index value in B obtained from mixture of two lognormlas and Black-Scholes lognormal methods for the period between December 23 2002 and December 18 2003. Critical value is 0.0861865.



A. Plot of Test Statistic D Value $Max\left|F_{VI}(x)-F_{2LN}(x)\right|$



B. Plot of $\left|F_{VI}(x)-F_{2LN}(x)\right|$

Figure 15: Kolmogorov-Smirnov Test Statistic of Risk-Neutral Density Estimates between Volatility Interpolation and Mixture of Two Lognormals Methods. Plot of the largest absolute deviation of the cumulative distribution functions in A and the absolute deviation across index value in B obtained from volatility interpolation and mixture of two lognormals methods for the period between December 23 2002 and December 18 2003. Critical value is 0.0861865.

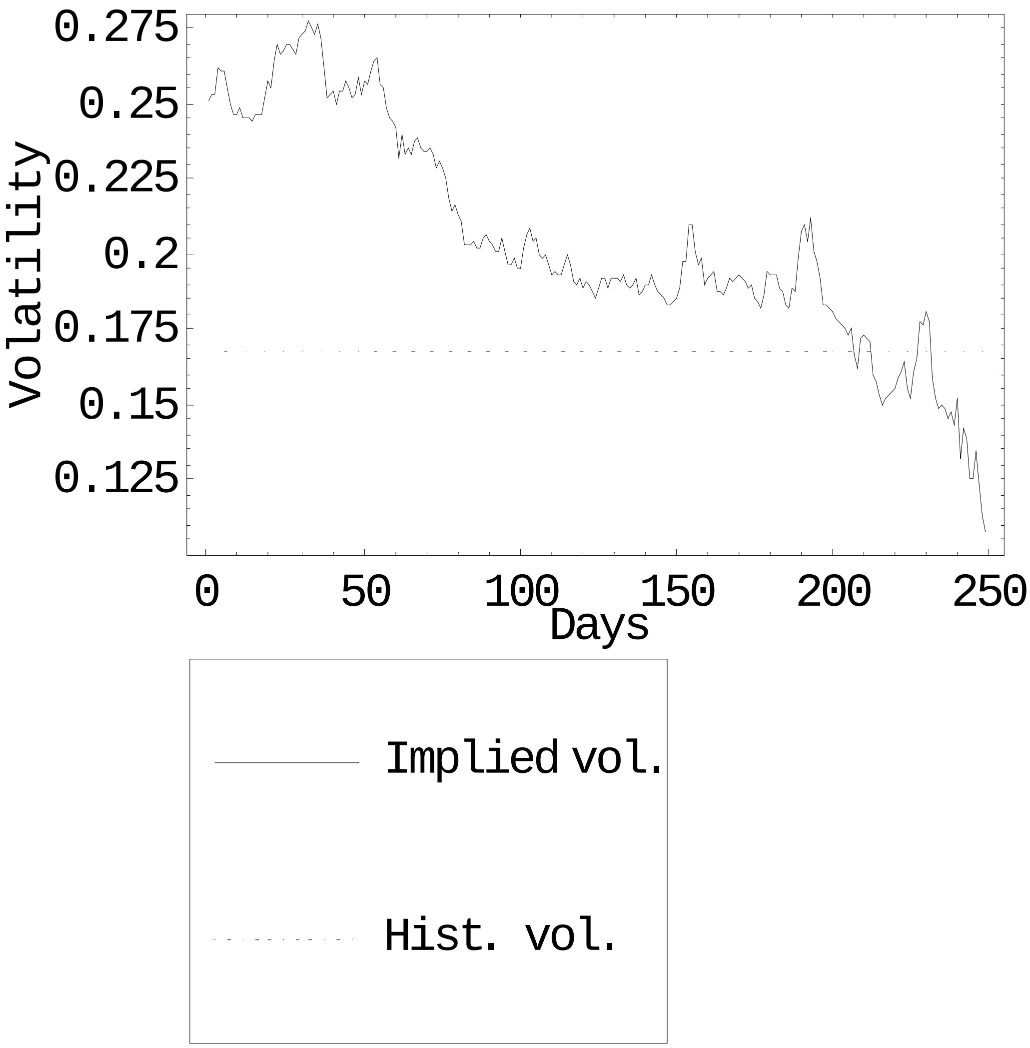


Figure 16: Historical Volatility and At the Money Implied Volatility. Plot of the historical volatility used to obtain the benchmark Black-Scholes lognormal dynamics and at the money implied volatilities incorporated in the option implied dynamics for the sample period between December 23 2002 and December 18 2003.

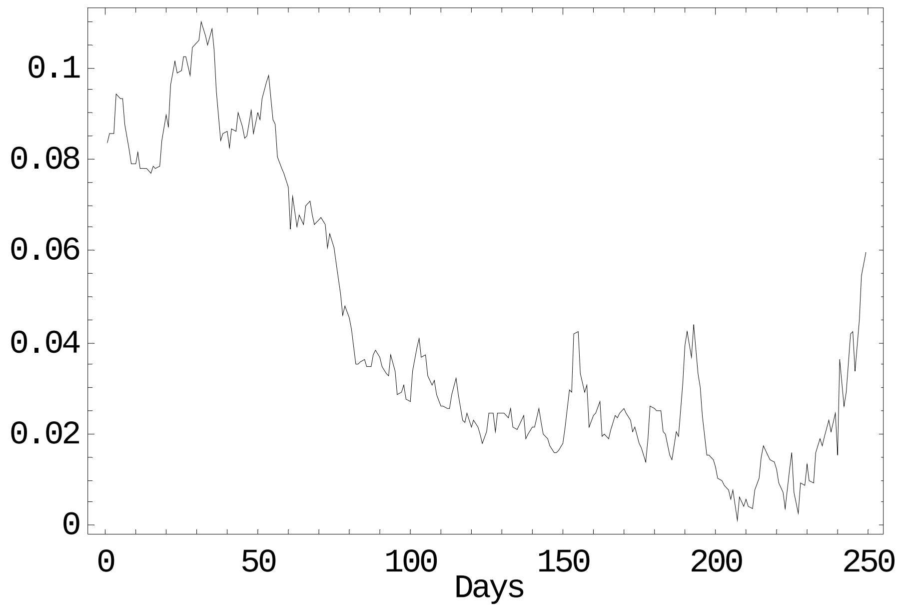


Figure 17: Absolute Deviation between Historical Volatility and At the Money Implied Volatility. Plot of the absolute value of the difference between historical volatilities used to obtain the benchmark Black-Scholes lognormal dynamics and at the money implied volatilities incorporated in the option implied dynamics for the sample period between December 23 2002 and December 18 2003.

Mathematical Appendix

A. Derivation of Closed form Option Pricing Formula (10) and (11) in a Mixture of Two Lognormals Model of Bahra (1997)

Under Black-Scholes assumptions, Cox, Ross, and Rubinstein (1979) show that the price of European call and put options is the present value of the expected future payoffs:

$$\text{call}(K, \tau = T - t) = e^{-r\tau} \int_K^{\infty} (S_T - K)\, f(S_T) dS_T \tag{a1}$$

$$\text{put}(K, \tau = T - t) = e^{-r\tau} \int_0^{K} (K - S_T)\, f(S_T) dS_T \tag{a2}$$

Risk-neutral density (RND) of the stock price at date T , $f(S_T)$ is directly specified as a mixture of two lognormals:

$$f(S_T) = \pi L(\mu_1, \sigma_1; S_T) + (1 - \pi) L(\mu_2, \sigma_2; S_T) \tag{a3}$$

where $L(\mu_i, \sigma_i; S_T) = \dfrac{1}{S_T \sigma_i \sqrt{2\pi}} \exp[-\dfrac{(\ln S_T - \mu_i)^2}{2\sigma_i^2}].$ (a4)

Substitution of (a3) into (a1) produces the following call option price function:

$$\text{call}(K, \tau) = e^{-r\tau} \int_K^{\infty} \left[\{\pi L(\mu_1, \sigma_1; S_T) + (1 - \pi) L(\mu_2, \sigma_2; S_T)\right] (S_T - K) dS_T \tag{a5}$$

Separate equation (a5) into two integrals:

$$\begin{aligned} \text{call}(K, \tau) &= e^{-r\tau} \int_K^{\infty} \left[\pi L(\mu_1, \sigma_1; S_T) + (1 - \pi) L(\mu_2, \sigma_2; S_T)\right] (S_T) dS_T \\ &\quad - e^{-r\tau} \int_K^{\infty} \left[\pi L(\mu_1, \sigma_1; S_T) + (1 - \pi) L(\mu_2, \sigma_2; S_T)\right] (K) dS_T \\ &\equiv A - B \end{aligned} \tag{a6}$$

Component A:

$$A = e^{-r\tau} \int_K^{\infty} \left[\pi L(\mu_1, \sigma_1; S_T) + (1 - \pi) L(\mu_2, \sigma_2; S_T)\right] (S_T) dS_T$$

$$= e^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_K^{\infty}\left[\frac{\pi}{\sigma_1}\exp[-\frac{\left(\ln S_T-\mu_1\right)^2}{2\sigma_1{}^2}]+\frac{1-\pi}{\sigma_2}\exp[-\frac{\left(\ln S_T-\mu_2\right)^2}{2\sigma_2{}^2}]\right]dS_T \qquad \text{(a7)}$$

Use a technique of a change of a variable, $v=\ln S_T$,

$$S_T = e^v \text{ and } dS_T = e^v dv \qquad \text{(a8)}$$

$$A = e^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_{\ln K}^{\infty}\left[\frac{\pi}{\sigma_1}\exp[-\frac{\left(v-\mu_1\right)^2}{2\sigma_1{}^2}]+\frac{1-\pi}{\sigma_2}\exp[-\frac{\left(v-\mu_2\right)^2}{2\sigma_2{}^2}]\right]e^v dv$$

$$= e^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_{\ln K}^{\infty}\left[\frac{\pi}{\sigma_1}\exp[v-\frac{\left(v-\mu_1\right)^2}{2\sigma_1{}^2}]+\frac{1-\pi}{\sigma_2}\exp[v-\frac{\left(v-\mu_2\right)^2}{2\sigma_2{}^2}]\right]dv \qquad \text{(a9)}$$

Use a technique of completing a square for the exponential terms:

$$v-\frac{\left(v-\mu_i\right)^2}{2\sigma_i{}^2} = \frac{2\sigma_i{}^2 v-(v^2-2v\mu_i+\mu_i{}^2)}{2\sigma_i{}^2}$$

$$= \frac{-(v^2-2v\mu_i+\mu_i{}^2-2\sigma_i{}^2 v)}{2\sigma_i{}^2} = \frac{-\{v^2-2(\mu_i+\sigma_i{}^2)v+\mu_i{}^2\}}{2\sigma_i{}^2}$$

$$= \frac{-\{v-(\mu_i+\sigma_i{}^2)\}^2+(\mu_i+\sigma_i{}^2)^2-\mu_i{}^2}{2\sigma_i{}^2} = \frac{-\{v-(\mu_i+\sigma_i{}^2)\}^2+\mu_i{}^2+2\mu_i\sigma_i{}^2+\sigma_i{}^4-\mu_i{}^2}{2\sigma_i{}^2}$$

$$= \frac{-\{v-(\mu_i+\sigma_i{}^2)\}^2}{2\sigma_i{}^2}+\mu_i+\frac{1}{2}\sigma_i{}^2 \qquad \text{(a10)}$$

Term A is now expressed as a mixture of two normal distributions:

$$A = e^{-r\tau}\frac{1}{\sqrt{2\pi}}[\frac{\pi}{\sigma_1}\exp(\mu_1+\frac{1}{2}\sigma_1{}^2)\int_{\ln K}^{\infty}\exp(\frac{-\{v-(\mu_1+\sigma_1{}^2)\}^2}{2\sigma_1{}^2})dv$$

$$+\frac{1-\pi}{\sigma_2}\exp(\mu_2+\frac{1}{2}\sigma_2{}^2)\int_{\ln K}^{\infty}\exp(\frac{-\{v-(\mu_2+\sigma_2{}^2)\}^2}{2\sigma_2{}^2})dv] \qquad \text{(a11)}$$

Perform a second change of variable $z_i = \dfrac{v-(\mu_i+\sigma_i{}^2)}{\sigma_i}$:

$$v = \sigma_i z_i + \mu_i + \sigma_i{}^2$$

$$dv = \sigma_1 dz_1 = \sigma_2 dz_2 \qquad \text{(a12)}$$

Thus, the term A is now expressed as a mixture of two standard normal distributions:

$$A = e^{-r\tau}\frac{1}{\sqrt{2\pi}}[\frac{\pi}{\sigma_1}\exp(\mu_1 + \frac{1}{2}\sigma_1^{\ 2})\int_{\frac{\ln K-(\mu_1+\sigma_1^{\ 2})}{\sigma_1}}^{\infty}\exp(-\frac{1}{2}z_1^{\ 2})\sigma_1 dz_1$$

$$+\frac{1-\pi}{\sigma_2}\exp(\mu_2 + \frac{1}{2}\sigma_2^{\ 2})\int_{\frac{\ln K-(\mu_2+\sigma_2^{\ 2})}{\sigma_2}}^{\infty}\exp(-\frac{1}{2}z_2^{\ 2})\sigma_2 dz_2]$$

$$= e^{-r\tau}[\pi\exp(\mu_1 + \frac{1}{2}\sigma_1^{\ 2})\int_{\frac{\ln K-(\mu_1+\sigma_1^{\ 2})}{\sigma_1}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z_1^{\ 2})dz_1$$

$$+(1-\pi)\exp(\mu_2 + \frac{1}{2}\sigma_2^{\ 2})\int_{\frac{\ln K-(\mu_2+\sigma_2^{\ 2})}{\sigma_2}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z_2^{\ 2})dz_2] \qquad \text{(a13)}$$

Let $Normal(x)$ be the cumulative density function of a standard normal distribution. Use the symmetric nature of the normal distribution:

$$\int_{a}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)dz = \int_{-\infty}^{-a}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)dz$$

Therefore, the term A becomes:

$$A = e^{-r\tau}[\pi\exp(\mu_1 + \frac{1}{2}\sigma_1^{\ 2})\int_{-\infty}^{-\frac{\ln K-(\mu_1+\sigma_1^{\ 2})}{\sigma_1}}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z_1^{\ 2})dz_1$$

$$+(1-\pi)\exp(\mu_2 + \frac{1}{2}\sigma_2^{\ 2})\int_{-\infty}^{-\frac{\ln K-(\mu_2+\sigma_2^{\ 2})}{\sigma_2}}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z_2^{\ 2})dz_2]$$

$$A = e^{-r\tau}[\pi\exp(\mu_1 + \frac{1}{2}\sigma_1^{\ 2})N(\frac{-\ln K + (\mu_1 + \sigma_1^{\ 2})}{\sigma_1})$$

$$+(1-\pi)\exp(\mu_2 + \frac{1}{2}\sigma_2^{\ 2})N(\frac{-\ln K + (\mu_2 + \sigma_2^{\ 2})}{\sigma_2})] \qquad \text{(a14)}$$

Component B:

$$B = e^{-r\tau}\int_{K}^{\infty}\left[\pi L\left(\mu_1,\sigma_1;S_T\right) + \left(1-\pi\right)L\left(\mu_2,\sigma_2;S_T\right)\right]\left(K\right)dS_T$$

$$B = e^{-r\tau} K \frac{1}{\sqrt{2\pi}} \int_K^\infty \left[\frac{\pi}{\sigma_1} \frac{1}{S_T} \exp(-\frac{\left(\ln S_T - \mu_1\right)^2}{2\sigma_1{}^2}) + \frac{1-\pi}{\sigma_2} \frac{1}{S_T} \exp(-\frac{\left(\ln S_T - \mu_2\right)^2}{2\sigma_2{}^2}) \right] dS_T \qquad \text{(a15)}$$

Perform a change of variable, $u_i = \dfrac{\ln S_T - \mu_i}{\sigma_i}$.

$$\ln S_T = \sigma_i u_i + \mu_i$$
$$S_T = e^{\sigma_i u_i + \mu_i}$$
$$dS_T = \frac{\partial e^{\sigma_i u_i + \mu_i}}{\partial u_i} du_i = \sigma_i e^{\sigma_i u_i + \mu_i} du_i = \sigma_i S_T du_i$$
$$dS_T = \sigma_1 S_T du_1 = \sigma_2 S_T du_2 \qquad \text{(a16)}$$

$$B = e^{-r\tau} K \frac{1}{\sqrt{2\pi}} [\frac{\pi}{\sigma_1} \int_{\frac{\ln K - \mu_1}{\sigma_1}}^\infty \frac{1}{S_T} \exp(-\frac{1}{2} u_1{}^2) \sigma_1 S_T du_1 + \frac{1-\pi}{\sigma_2} \int_{\frac{\ln K - \mu_2}{\sigma_2}}^\infty \frac{1}{S_T} \exp(-\frac{1}{2} u_2{}^2) \sigma_2 S_T du_2]$$
$$B = e^{-r\tau} K \frac{1}{\sqrt{2\pi}} [\pi \int_{\frac{\ln K - \mu_1}{\sigma_1}}^\infty \exp(-\frac{1}{2} u_1{}^2) du_1 + (1-\pi) \int_{\frac{\ln K - \mu_2}{\sigma_2}}^\infty \exp(-\frac{1}{2} u_2{}^2) du_2]$$
$$B = e^{-r\tau} K [\pi \int_{\frac{\ln K - \mu_1}{\sigma_1}}^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} u_1{}^2) du_1 + (1-\pi) \int_{\frac{\ln K - \mu_2}{\sigma_2}}^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} u_2{}^2) du_2]$$

Using $Normal(x)$ for the cumulative density function of a standard normal distribution and due to the symmetric nature of the normal distribution B is now represented as:

$$B = e^{-r\tau} K [\pi \int_{-\infty}^{-\frac{\ln K - \mu_1}{\sigma_1}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} u_1{}^2) du_1 + (1-\pi) \int_{-\infty}^{-\frac{\ln K - \mu_2}{\sigma_2}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} u_2{}^2) du_2]$$
$$B = e^{-r\tau} K [\pi N(\frac{-\ln K + \mu_1}{\sigma_1}) + (1-\pi) N(\frac{-\ln K + \mu_2}{\sigma_2})] \qquad \text{(a17)}$$

Substitution of (a14) and (a16) into (a6) yields the closed form solution:

$$\text{call}\left(K, \tau\right) \equiv A - B$$
$$= e^{-r\tau} [\pi \exp(\mu_1 + \frac{1}{2}\sigma_1{}^2) N(\frac{-\ln K + (\mu_1 + \sigma_1{}^2)}{\sigma_1}) + (1-\pi) \exp(\mu_2 + \frac{1}{2}\sigma_2{}^2) N(\frac{-\ln K + (\mu_2 + \sigma_2{}^2)}{\sigma_2})]$$
$$- e^{-r\tau} K [\pi N(\frac{-\ln K + \mu_1}{\sigma_1}) + (1-\pi) N(\frac{-\ln K + \mu_2}{\sigma_2})]$$

$$\text{call}\left(K,\tau\right)=e^{-r\tau}\pi[\exp(\mu_1+\frac{1}{2}\sigma_1^{\,2})N(\frac{-\ln K+(\mu_1+\sigma_1^{\,2})}{\sigma_1})-KN(\frac{-\ln K+\mu_1}{\sigma_1})]$$

$$+e^{-r\tau}(1-\pi)[\exp(\mu_2+\frac{1}{2}\sigma_2^{\,2})N(\frac{-\ln K+(\mu_2+\sigma_2^{\,2})}{\sigma_2})-KN(\frac{-\ln K+\mu_2}{\sigma_2})] \qquad \text{(a18)}$$

Finally,

$$\text{call}\left(K,\tau\right)=e^{-r\tau}[\pi\{\exp(\mu_1+\frac{1}{2}\sigma_1^{\,2})N(d_1)-KN(d_2)\}$$

$$+(1-\pi)\{\exp(\mu_2+\frac{1}{2}\sigma_2^{\,2})N(d_3)-KN(d_4)\}] \qquad \text{(a19)}$$

where $d_1=\frac{-\ln K+(\mu_1+\sigma_1^{\,2})}{\sigma_1}$, $d_2=d_1-\sigma_1$, $d_3=\frac{-\ln K+(\mu_2+\sigma_2^{\,2})}{\sigma_2}$, and $d_4=d_3-\sigma_2$.

Closed form solution to the European put option pricing function can be obtained in a similar manner.

B. Derivation of Fokker-Planck (Forward Kolmogorov) Equation[r]

A stochastic process $\xi(t)$ is said to be continuous if there is only a small probability that $\xi(t)$ will take on an appreciable increment in a short interval of time. This means that for any positive constant δ:

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x_T-x_t|\geq\delta} d_{x_T}F(x_T,T;x_t,t)=0 \qquad \text{(a20)}$$

where $\Delta t=T-t$. Make the following assumptions:

1) The partial derivatives $\frac{\partial F(x_T,T;x_t,t)}{\partial x_t}$ and $\frac{\partial^2 F(x_T,T;x_t,t)}{\partial x_t^{\,2}}$ exist and are continuous for arbitrary values of t, $T>t$, x_t, and x_T.
2) For any $\delta>0$, the following limits exist and the convergence is uniform in x_t.[s]

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x_T-x_t|<\delta}(x_T-x_t)d_{x_T}F(x_T,T;x_t,t)=a(t,x_t) \qquad \text{(a21)}$$

[r] This is based on Gnedenko.

[s] The left –hands sides of equations (a21) and (a22) depend on δ.

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x_T - x_t| < \delta} (x_T - x_t)^2 d_{x_T} F(x_T, T; x_t, t) = b^2(t, x_t) \qquad \text{(a22)}$$

3) A probability density function exists:

$$f(x_T, T; x_t, t) = \frac{\partial F(x_T, T; x_t, t)}{\partial x_T} \qquad \text{(a23)}$$

4) The following derivatives exist and are continuous:

$$\frac{\partial f(x_T, T; x_t, t)}{\partial T}$$
$$\frac{\partial}{\partial x_T}[a(T, x_T) f(x_T, T; x_t, t)] \qquad \text{(a24)}$$
$$\frac{\partial^2}{\partial {x_T}^2}[b(T, x_T) f(x_T, T; x_t, t)]$$

THE FOKKER-PLANCK EQUATION (FORWARD KOLMOGOROV EQUATION):
For any continuous stochastic process without after effect satisfying conditions 1) through 4), the probability density function $f(x_T, T; x_t, t)$ *is a solution of the equation:*

$$\frac{\partial f(x_T, T; x_t, t)}{\partial T} = -\frac{\partial (a(x_T, T) f(x_T, T; x_t, t))}{\partial x_T} + \frac{1}{2}\frac{\partial^2 (b(x_T, T)^2 f(x_T, T; x_t, t))}{\partial {X_T}^2} \qquad \text{(a25)}$$

Derivation: Let c and d ($c < d$) denote certain numbers and $R(x_T)$ a nonnegative continuous function having continuous first- and second-order derivatives. Assume:

$$R(x_T) = 0 \quad \text{for} \quad x_T < c \quad \text{and} \quad x_T > d\,.$$

Due to the condition that the function $R(x_T)$ and its derivatives are continuous,

$$R(c) = R(d) = R'(c) = R'(d) = R''(c) = R''(d) \qquad \text{(a26)}$$

Note:

$$\int_c^d \frac{\partial f(x_T, T; x_t, t)}{\partial T} R(x_T) dx_T = \frac{\partial}{\partial T}\int_c^d f(x_T, T; x_t, t) R(x_T) dx_T$$
$$= \lim_{\Delta T \to 0} \int \frac{f(x_T, T + \Delta T; x_t, t) - f(x_T, T; x_t, t)}{\Delta T} R(x_T) dx_T \qquad \text{(a27)}$$

Apply the generalized Markov equation:

$$f(x_T, T+\Delta T; x_t, t) = \int f(z, T; x_t, t) f(x_T, T+\Delta T; z, T)$$

Thus, the equation (a27) can be written down as:

$$\begin{aligned}
&\int_c^d \frac{\partial f(x_T, T; x_t, t)}{\partial T} R(x_T) dx_T \\
&= \lim_{\Delta T \to 0} \frac{1}{\Delta T} [\iint f(z, T; x_t, t) f(x_T, T+\Delta T; z, T) R(x_T) dz dx_T - \int f(x_T, T; x_t, t) R(x_T) dx_T] \\
&= \lim_{\Delta T \to 0} \frac{1}{\Delta T} [\int f(z, T; x_t, t) \int f(x_T, T+\Delta T; z, T) R(x_T) dx_T dz - \int f(x_T, T; x_t, t) R(x_T) dx_T] \\
&= \lim_{\Delta T \to 0} \frac{1}{\Delta T} \int f(x_T, T; x_t, t) [\int f(z, T+\Delta T; x_T, T) R(z) dz - R(x_T)] dx_T \qquad \text{(a28)}
\end{aligned}$$

First, the order of integration is interchanged. Second, the notation for the variables of integration is changed (Replace x_T by z and z by x_T).

By Taylor's theorem:

$$R(z) = R(x_T) + (z - x_T) R'(x_T) + \frac{1}{2}(z - x_T)^2 R''(x_T) + o[(z - x_T)^2]$$

By the bounded nature of the function $R(z)$ and the condition 1):

$$\int_{|x_T - z| \geq \delta} f(z, T+\Delta T; x_T, T) R(z) dz = o(\Delta T)$$

$$\int_{|x_T - z| \leq \delta} f(z, T+\Delta T; x_T, T) R(z) dz = 1 + o(\Delta T)$$

It follows that:

$$\int f(z, T+\Delta T; x_T, T) R(z) dz - R(x_T) = R'(x_T) \int_{|x_T - z| < \delta} (z - y) \int f(z, T+\Delta T; x_T, T) dz$$

$$+ \frac{1}{2} R''(x_T) \int_{|x_T - z| < \delta} [(z - x_T)^2 + o((z - x_T)^2)] f(z, T+\Delta T; x_T, T) dz + o(\Delta T).$$

Thus,

$$\int_c^d \frac{\partial f(x_T, T; x_t, t)}{\partial T} R(x_T) dx_T =$$

$$\lim_{\Delta T \to 0} \frac{1}{\Delta T} \int f(x_T, T; x_t, t)\{R'(x_T) \int_{|x_T - z| < \delta} (z - y) f(z, T + \Delta T; x_T, T) dz$$

$$+ \frac{1}{2} R''(x_T) \int_{|x_T - z| < \delta} [(z - x_T)^2 + o((z - x_T)^2)] f(z, T + \Delta T; x_T, T) dz + o(\Delta T)\} dx_T \,.$$

Let $\Delta t \to 0$. Following the assumption that the limits in 2) and 3) are uniform in x_t, the limit equation of the above can be expressed as:

$$\int_c^d \frac{\partial f(x_T, T; x_t, t)}{\partial T} R(x_T) dx_T = \int f(x_T, T; x_t, t)[a(T, x_T) R'(x_T) + \frac{1}{2} b^2(T, x_T) R''(x_T)] dx_T$$

Because $R'(x_T) = R''(x_T) = 0$ for $x_T \le c$ and $x_T \ge d$:

$$\int_c^d \frac{\partial f(x_T, T; x_t, t)}{\partial T} R(x_T) dx_T = \int_c^d f(x_T, T; x_t, t)[a(T, x_T) R'(x_T) + \frac{1}{2} b^2(T, x_T) R''(x_T)] dx_T \qquad \text{(a29)}$$

Use integration by parts and the equation (a26),

$$\int_c^d f(x_T, T; x_t, t) a(T, x_T) R'(x_T) dx_T = -\int_c^d R(x_T) \frac{\partial}{\partial x_T} [a(T, x_T) f(x_T, T; x_t, t)] dx_T$$

$$\int_c^d f(x_T, T; x_t, t) b^2(T, x_T) R''(x_T) dx_T = \int_c^d R(x_T) \frac{\partial^2}{\partial {x_T}^2} [b^2(T, x_T) f(x_T, T; x_t, t)] dx_T$$

Substitute these expressions to the equation (a29):

$$\int_c^d \frac{\partial f(x_T, T; x_t, t)}{\partial T} R(x_T) dx_T$$

$$= \int_c^d \{ -\frac{\partial}{\partial x_T} [a(T, x_T) f(x_T, T; x_t, t)] + \frac{1}{2} \frac{\partial^2}{\partial {x_T}^2} [b^2(T, x_T) f(x_T, T; x_t, t)] \} R(x_T) dx_T \,.$$

This equation can be written in the following form:

$$\int_c^d \{ \frac{\partial f(x_T, T; x_t, t)}{\partial T} + \frac{\partial}{\partial x_T} [a(T, x_T) f(x_T, T; x_t, t)]$$

$$- \frac{1}{2} \frac{\partial^2}{\partial {x_T}^2} [b^2(T, x_T) f(x_T, T; x_t, t)] \} R(x_T) dx_T = 0 \qquad \text{(a30)}$$

Fokker-Planck equation follows from (a30) since the function $R(x_T)$ is arbitrary.

C. Barone-Adesi and Whaley (1987) Quadratic Approximation Method to Adjust for the Early Exercise Premium[t]

Consider a portfolio P of the one long (European or American) option position $V(S,t)$ on a stock S with continuously compounded dividend yield q written at time t and a short position of the stock in quantity Δ:

$$P = V(S,t) - \Delta S \qquad \text{(a20)}$$

The underlying asset price dynamics follows usual geometric Brownian motion:

$$dS = (\mu - q)Sdt + \sigma SdW \qquad \text{(a21)}$$

Portfolio value changes by

$$dP = dV - \Delta dS \qquad \text{(a22)}$$

From Ito's lemma, the change in the value of the option is written as

$$dV = (\frac{\partial V}{\partial S}(\mu - q)S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial V}{\partial S}\sigma SdW \qquad \text{(a23)}$$

Substitution of (a21) and (a23) into (a22) gives the change in the portfolio value as:

$$dP = (\frac{\partial V}{\partial S}(\mu - q)S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial V}{\partial S}\sigma SdW - \Delta\{(\mu - q)Sdt + \sigma SdW\}$$

After rearrangement:

$$dP = [(\frac{\partial V}{\partial S} - \Delta)(\mu - q)S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2]dt + (\frac{\partial V}{\partial S} - \Delta)\sigma SdW \qquad \text{(a24)}$$

Choosing $\Delta = \frac{\partial V}{\partial S}$ makes the portfolio risk-free since randomness dW is eliminated:

$$dP = (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt \qquad \text{(a25)}$$

[t] This is based on Hull and Shaw.

In an infinitesimal time interval dt, the portfolio holder earns capital gains equal to dP and loses dividends on the stock position (since the portfolio holder has short position in stock):

$$qS\frac{\partial V}{\partial S}dt \qquad \text{(a26)}$$

This portfolio is expected to grow at the risk-free interest rate r:

$$(\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt - qS\frac{\partial V}{\partial S}dt = rPdt \qquad \text{(a27)}$$

Substitution of (a20) into (a27) yields Black-Scholes PDE:

$$\frac{\partial V}{\partial t}+(r-q)S\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}=rV \qquad \text{(a28)}$$

Let ρ denote an early exercise premium which is the difference between American style and European style option price written on the same underlying with same maturity. ρ satisfies the Black-Scholes PDE because both American style and European style option price satisfy it:

$$\frac{\partial \rho}{\partial t}+(r-q)S\frac{\partial \rho}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 \rho}{\partial S^2}=r\rho \qquad \text{(a29)}$$

The "quasi-stationary method" is a standard method for obtaining approximate solutions to differential equations of this type. Apply a change of variable technique of the following, set $\rho = h(\tau)g(S,h)$, and ignore the time-dependence ($\partial\rho/\partial t = 0$):

$$\tau = T-t,\ h(\tau)=1-e^{-r\tau},\ k_1=\frac{2r}{\sigma^2},\ k_2=\frac{2(r-q)}{\sigma^2}$$

Equation (a29):

$$\frac{1}{2}S^2\frac{\partial^2 \rho}{\partial S^2}\sigma^2 - r\rho + (r-q)S\frac{\partial \rho}{\partial S}=0$$

can now be written down as:

$$S^2\frac{\partial^2 g}{\partial S^2}+k_2 S\frac{\partial g}{\partial S}-\frac{k_1 g}{h}-(1-h)k_1\frac{\partial g}{\partial h}=0 \qquad \text{(a30)}$$

With quasi-stationary approximation $(1-h)k_1 \frac{\partial g}{\partial h} = 0$:[u]

$$S^2 \frac{\partial^2 g}{\partial S^2} + k_2 S \frac{\partial g}{\partial S} - \frac{k_1 g}{h} = 0 \qquad \text{(a31)}$$

Equation (a31) is an equation of homogeneous type, which is easily solved in terms of a power of S .[v] Let $C_{Ameri}(S,t)$ and $P_{Ameri}(S,t)$ denote the American style call and put option prices. Let $C_{Euro}(S,t)$ and $P_{Euro}(S,t)$ denote the European style call and put option prices. After applying boundary conditions[w] to a solution of equation (a31):

$$C_{Ameri}(S,t) = \begin{cases} C_{Euro}(S,t) + A_2 (\frac{S}{B^*})^{\gamma_2} & \text{if} \quad S < B^* \\ S - K & \text{if} \quad S \geq B^* \end{cases} \qquad \text{(a32)}$$

$$P_{Ameri}(S,t) = \begin{cases} P_{Euro}(S,t) + A_1 (\frac{S}{B^{**}})^{\gamma_1} & \text{if} \quad S > B^{**} \\ K - S & \text{if} \quad S \leq B^{**} \end{cases} \qquad \text{(a33)}$$

B^* is the critical stock price above which the call option should be exercised and B^{**} is the critical stock price below which the put option should be exercised. These can be estimated by solving the following equations:

$$B^* - K = C_{Euro}(B^*, t) + [1 - e^{-q(T-t)} N(d_1(B^*))] \frac{B^*}{\gamma_2} \qquad \text{(a34)}$$

$$K - B^{**} = P_{Euro}(B^{**}, t) - [1 - e^{-q(T-t)} N(-d_1(B^{**}))] \frac{B^{**}}{\gamma_1} \qquad \text{(a35)}$$

Note in above equations:

$$\gamma_1 = \frac{1}{2}(1 - K_2 - \sqrt{(1-K_2)^2 + \frac{4K_1}{h}})$$

$$\gamma_2 = \frac{1}{2}(1 - K_2 + \sqrt{(1-K_2)^2 + \frac{4K_1}{h}})$$

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[u] When τ is large, $1-h$ is close to zero. When τ is small, $\partial g / \partial h$ is close to zero.

[v] The solution is $g = A_1 (S / B_t)^{\gamma_1}$.

[w] For call, $\lim_{S \to 0} \rho(S,t) = 0$. For put, $\lim_{S \to \infty} \rho(S,t) = 0$.

$$A_1 = -(\frac{B^{**}}{\gamma_1})[1 - e^{-q(T-t)} N(-d_1(B^{**}))] \qquad \text{(a36)}$$

$$A_2 = (\frac{B^{*}}{\gamma_2})[1 - e^{-q(T-t)} N(d_1(B^{*}))]$$

$$d_1(S) = \frac{\ln(S/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

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